On negative eigenvalues of two-dimensional Schrödinger operators

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Abstract

The paper presents estimates for the number of negative eigenvalues of a two-dimensional Schrödinger operator in terms of $L \log L$ type Orlicz norms of the potential and proves a conjecture by N.N. Khuri, A. Martin and T.T. Wu.

1 Introduction

According to the celebrated Cwikel-Lieb-Rozenblum inequality, the number $N_-(V)$ of negative eigenvalues of the Schrödinger operator $-\Delta - V$, $V \geq 0$ on $L_2(\mathbb{R}^d)$, $d \geq 3$ is estimated above by

$$\text{const} \int_{\mathbb{R}^d} V(x)^{d/2} dx.$$

It is well known that this estimate does not hold for $d = 2$. In this case, the Schrödinger operator has at least one negative eigenvalue for any nonzero $V \geq 0$, and no estimate of the type

$$N_-(V) \leq \text{const} + \int_{\mathbb{R}^2} V(x) W(x)\, dx$$

can hold, provided the weight function W is bounded in a neighborhood of at least one point (see [10]). On the other hand,

$$N_-(V) \geq \text{const} \int_{\mathbb{R}^2} V(x)\, dx$$

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(see [11]). Upper estimates for $N_-(V)$ in the case $d = 2$ can be found in [4, 6, 7, 10, 17, 20, 21, 25, 26, 29, 30] and in the references therein. Following the pioneering paper [29], we obtain upper estimates involving $L \log L$ type Orlicz norms of the potential (Theorems 3.1, 6.1 anf 7.1) and prove (Theorem 4.3) a conjecture by N.N. Khuri, A. Martin and T.T. Wu ([17]). In fact, we show that the main estimate in [29] is actually stronger than the one conjectured in [17] (see Section 8 below). Our approach does not seem to be sufficient to settle the stronger conjecture by K. Chadan, N.N. Khuri, A. Martin and T.T. Wu ([7]; see (23), (24)).
We discuss several upper estimates for $N_-(V)$ in the paper and show in Section 8 how they are related to each other (see the diagram close to the end of Section 8). All of them involve terms of two types: integrals of V with a logarithmic weight and $L \log L$ type (or L_p, $p > 1$) norms of V. None of these estimates is sharp in the sense that $N_-(V)$ has to be infinite if the right-hand side is infinite. On the other hand, both the logarithmic weight and the $L \log L$ norm are in a sense optimal if one tries to estimate $N_-(V)$ in terms of weighted integrals of V and of its Orlicz norms (see [29, Section 4] and Section 9 below). It is probably difficult to obtain an estimate for $N_-(V)$ that is sharp in the above sense. Indeed, there are potentials $V \geq 0$ such that $N_-(\alpha V) < \infty$ for $\alpha < 1$ and $N_-(\alpha V) = \infty$ for $\alpha > 1$. For such potentials, $N_-(V)$ may be finite or infinite, and in the latter case, $N_-(\alpha V)$ may grove arbitrarily fast or arbitrarily slow as $\alpha \to 1 - 0$ (see Theorem 9.3 and Appendix B).

2 Notation and auxiliary results

We need some notation from the theory of Orlicz spaces (see [18, 28]). Let (Ω, Σ, μ) be a measure space, let Φ and Ψ be mutually complementary N-functions, and let $L_\Phi(\Omega)$, $L_\Psi(\Omega)$ be the corresponding Orlicz spaces. (These spaces are denoted by $L^*_\Phi(\Omega)$, $L^*_\Psi(\Omega)$ in [18], where Ω is assumed to be a closed bounded subset of $\mathbb{R}^d$ equipped with the standard Lebesgue measure.) We will use the following norms on $L_\Psi(\Omega)$

$$\|f\|_\Psi = \|f\|_{\Psi,\Omega} = \sup\left\{\left|\int_\Omega fg d\mu\right| : \int_\Omega \Phi(g) d\mu \leq 1\right\} \tag{1}$$

and

$$\|f\|_{(\Psi)} = \|f\|_{(\Psi,\Omega)} = \inf\left\{\kappa > 0 : \int_\Omega \Psi\left(\frac{f}{\kappa}\right) d\mu \leq 1\right\}. \tag{2}$$

These two norms are equivalent

$$\|f\|_{(\Psi)} \leq \|f\|_\Psi \leq 2\|f\|_{(\Psi)}, \quad \forall f \in L_\Psi(\Omega). \tag{3}$$

Note that

$$\int_\Omega \Psi\left(\frac{f}{\kappa_0}\right) d\mu \le C_0, \quad C_0 \ge 1 \implies \|f\|_{(\Psi)} \le C_0\kappa_0. \tag{4}$$

Indeed, since Ψ is even, convex and increasing on $[0, +\infty)$, and $\Psi(0) = 0$, we get for any $\kappa \ge C_0\kappa_0$,

$$\int_\Omega \Psi\left(\frac{f}{\kappa}\right) d\mu \le \int_\Omega \Psi\left(\frac{f}{C_0\kappa_0}\right) d\mu \le \frac{1}{C_0}\int_\Omega \Psi\left(\frac{f}{\kappa_0}\right) d\mu \le 1. \tag{5}$$

It follows from (4) with $\kappa_0 = 1$ that

$$\|f\|_{(\Psi)} \le \max\left\{1, \int_\Omega \Psi(f)d\mu\right\}. \tag{6}$$

We will need the following equivalent norm on $L_\Psi(\Omega)$ with $\mu(\Omega) < \infty$ which was introduced in [29]:

$$\|f\|_\Psi^{(\mathrm{av})} = \|f\|_{\Psi,\Omega}^{(\mathrm{av})} = \sup\left\{\left|\int_\Omega fgd\mu\right| : \int_\Omega \Phi(g)d\mu \le \mu(\Omega)\right\}. \tag{7}$$

Lemma 2.1.

$$\min\{1, \mu(\Omega)\}\, \|f\|_{\Psi,\Omega} \le \|f\|_{\Psi,\Omega}^{(\mathrm{av})} \le \max\{1, \mu(\Omega)\}\, \|f\|_{\Psi,\Omega}$$

Proof. Let

$$\mathbb{B}_1 := \left\{g : \int_\Omega \Phi(g)d\mu \le 1\right\}, \quad \mathbb{B}_\Omega := \left\{g : \int_\Omega \Phi(g)d\mu \le \mu(\Omega)\right\}.$$

Suppose $\mu(\Omega) \ge 1$. Then, clearly, $\|f\|_{\Psi,\Omega} \le \|f\|_{\Psi,\Omega}^{(\mathrm{av})}$. It is easy to see that

$$g \in \mathbb{B}_\Omega \implies \frac{1}{\mu(\Omega)}\, g \in \mathbb{B}_1$$

(cf. (5)). Hence

$$\|f\|_{\Psi,\Omega}^{(\mathrm{av})} = \sup_{g\in\mathbb{B}_\Omega}\left|\int_\Omega fgd\mu\right| \le \sup_{h\in\mathbb{B}_1}\left|\int_\Omega f\cdot\big(\mu(\Omega)h\big)\,d\mu\right| = \mu(\Omega)\|f\|_{\Psi,\Omega}.$$

Suppose now $\mu(\Omega) < 1$. Then $\|f\|_{\Psi,\Omega}^{(\mathrm{av})} \le \|f\|_{\Psi,\Omega}$ and

$$g \in \mathbb{B}_1 \implies \mu(\Omega)\, g \in \mathbb{B}_\Omega.$$

Hence,

$$\mu(\Omega)\|f\|_{\Psi,\Omega} = \mu(\Omega)\sup_{g\in\mathbb{B}_1}\left|\int_\Omega fgd\mu\right| \le \sup_{h\in\mathbb{B}_\Omega}\left|\int_\Omega fhd\mu\right| = \|f\|_{\Psi,\Omega}^{(\mathrm{av})}.$$

□

We will need the following pair of mutually complementary N-fuctions

$$\mathcal{A}(s) = e^{|s|} - 1 - |s|, \quad \mathcal{B}(s) = (1+|s|)\ln(1+|s|) - |s|, \quad s \in \mathbb{R}. \tag{8}$$

We will use the following standard notation

$$a_+ := \max\{0, a\}, \quad a \in \mathbb{R}. \tag{9}$$

Lemma 2.2. *$\frac{1}{2}\, s \ln_+ s \le \mathcal{B}(s) \le s + 2s \ln_+ s, \ \ \forall s \ge 0.$*

Proof. Integrating the inequality

$$1 + \ln s = \ln(es) < \ln(1+3s) \le 2\ln(1+s), \quad s \ge 1,$$

one gets $s \ln_+ s \le 2\mathcal{B}(s)$.
If $s \ge 1$, then

$$\begin{aligned}\mathcal{B}(s) = (1+s)\ln(1+s) - s \le 2s\ln(2s) - s = (2\ln 2 - 1)s + 2s \ln s \\ < s + 2s\ln s.\end{aligned}$$

If $s \in [0, 1)$, then integrating the inequality $\ln(1+s) \le s$ one gets

$$(1+s)\ln(1+s) - s \le \frac{1}{2}\, s^2 \le \frac{1}{2}\, s \le s.$$

□

Lemma 2.3. *$\mathcal{B}(s) \le \frac{1}{2}\, s^2, \ \ \frac{1}{2}\, s^2 \le \mathcal{A}(s) \le \frac{e}{2}\, s^2, \ \ \forall s \in [0, 1].$*

Proof. The first inequality was proved at the end of the proof of Lemma 2.2, the second one is obtained by integrating the inequality $1 \le e^s \le e, \ s \in [0, 1]$ twice. □

Lemma 2.4. *$e^s \le 2\mathcal{A}(s) + \frac{3}{2}, \ \ \forall s \ge 0.$*

Proof.

$$s \le \frac{1}{2} + \frac{s^2}{2} \le \frac{1}{2} + \mathcal{A}(s).$$

Hence

$$e^s = \mathcal{A}(s) + 1 + s \le 2\mathcal{A}(s) + \frac{3}{2}, \quad \forall s \ge 0.$$

□

Lemma 2.5. *Let $\mu(\Omega) > 1$. Then*

$$\|f\|^{(\mathrm{av})}_{\mathcal{B},\Omega} \le \|f\|_{\mathcal{B},\Omega} + \ln\left(\frac{7}{2}\,\mu(\Omega)\right)\, \|f\|_{L_1(\Omega,\mu)}.$$

Proof. Since $\|f\|^{(\mathrm{av})}_{\mathcal{B},\Omega} = \||f|\|^{(\mathrm{av})}_{\mathcal{B},\Omega}$ and $\|f\|_{\mathcal{B},\Omega} = \||f|\|_{\mathcal{B},\Omega}$, we can assume without loss of generality that $f \ge 0$. In this case,

$$\|f\|_{\mathcal{B},\Omega} = \sup\left\{\int_\Omega fg\,d\mu : \ g \ge 0, \ \int_\Omega \mathcal{A}(g)d\mu \le 1\right\},$$
$$\|f\|^{(\mathrm{av})}_{\mathcal{B},\Omega} = \sup\left\{\int_\Omega fg\,d\mu : \ g \ge 0, \ \int_\Omega \mathcal{A}(g)d\mu \le \mu(\Omega)\right\}.$$

Take an arbitrary $g \ge 0$ with $\int_\Omega \mathcal{A}(g)d\mu \le \mu(\Omega)$ and set

$$\chi(x) := \begin{cases} 1 & \text{if } g(x) \le \ln\left(\frac{7}{2}\,\mu(\Omega)\right), \\ \\ 0 & \text{if } g(x) > \ln\left(\frac{7}{2}\,\mu(\Omega)\right), \end{cases}$$
$$g_1 := g\chi + \ln\left(\tfrac{7}{2}\,\mu(\Omega)\right)\,(1-\chi), \quad g_2 := g - g_1 = \left(g - \ln\left(\tfrac{7}{2}\,\mu(\Omega)\right)\right)(1-\chi).$$

Then $g = g_1 + g_2$, $0 \le g_1 \le \ln\left(\frac{7}{2}\,\mu(\Omega)\right)$, and it follows from Lemma 2.4 that

$$\int_\Omega \mathcal{A}(g_2)d\mu = \int_{g(x)>\ln\left(\frac{7}{2}\,\mu(\Omega)\right)} \mathcal{A}\left(g(x) - \ln\left(\frac{7}{2}\,\mu(\Omega)\right)\right) d\mu(x)$$
$$\le \int_{g(x)>\ln\left(\frac{7}{2}\,\mu(\Omega)\right)} e^{g(x)-\ln\left(\frac{7}{2}\,\mu(\Omega)\right)} d\mu(x) \le \frac{2}{7\mu(\Omega)}\int_\Omega e^g d\mu$$
$$\le \frac{2}{7\mu(\Omega)}\int_\Omega\left(2\mathcal{A}(g) + \frac{3}{2}\right)d\mu \le \frac{2}{7\mu(\Omega)}\left(2\mu(\Omega) + \frac{3}{2}\,\mu(\Omega)\right) = 1.$$

Hence

$$\int_\Omega fg\,d\mu = \int_\Omega fg_1\,d\mu + \int_\Omega fg_2\,d\mu \le \ln\left(\frac{7}{2}\,\mu(\Omega)\right)\int_\Omega f\,d\mu + \|f\|_{\mathcal{B},\Omega}$$
$$= \|f\|_{\mathcal{B},\Omega} + \ln\left(\frac{7}{2}\,\mu(\Omega)\right)\,\|f\|_{L_1(\Omega,\mu)}.$$

□

3 A Solomyak type estimate

Let $\mathcal{H}$ be a Hilbert space and let $\mathbf{q}$ be a Hermitian form with a domain $\mathrm{Dom}\,(\mathbf{q}) \subseteq \mathcal{H}$. Set

$$N_-(\mathbf{q}) := \sup\left\{\dim\mathcal{L} \mid \mathbf{q}[u] < 0, \ \forall u \in \mathcal{L}\setminus\{0\}\right\}, \tag{10}$$

where $\mathcal{L}$ denotes a linear subspace of $\mathrm{Dom}\,(\mathbf{q})$. If $\mathbf{q}$ is the quadratic form of a self-adjoint operator A with no essential spectrum in $(-\infty, 0)$, then by

the variational principle, $N_-(\mathbf{q})$ is the number of negative eigenvalues of A repeated according to their multiplicity (see, e.g., [3, S1.3] or [5, Theorem 10.2.3]).
Let $V \geq 0$ be locally integrable on $\mathbb{R}^2$. Consider

$$\begin{gathered}\mathcal{E}_V[w] := \int_{\mathbb{R}^2} |\nabla w(x)|^2 dx - \int_{\mathbb{R}^2} V(x)|w(x)|^2 dx, \\ \mathrm{Dom}\,(\mathcal{E}_V) = W_2^1\left(\mathbb{R}^2\right) \cap L_2\left(\mathbb{R}^2, V(x)dx\right).\end{gathered}$$

Theorem 3.1. *There exists a constant $C > 0$ such that*

$$N_-(\mathcal{E}_V) \leq C\left(\|V\|_{\mathcal{B},\mathbb{R}^2} + \int_{\mathbb{R}^2} V(x)\ln(1+|x|)\,dx\right) + 1, \quad \forall V \geq 0. \tag{11}$$

Proof. Let

$$\begin{gathered}\mathcal{E}_{V,m}[w] := \int_{\Omega_m} |\nabla w(x)|^2 dx - \int_{\Omega_m} V(x)|w(x)|^2 dx, \\ \mathrm{Dom}\,(\mathcal{E}_{V,m}) = W_2^1\left(\Omega_m\right) \cap L_2\left(\Omega_m, V(x)dx\right), \quad m = 1, 2,\end{gathered}$$

where

$$\Omega_1 = B(0,1) = \{x \in \mathbb{R}^2 :\ |x| < 1\},\ \ \Omega_2 = \mathbb{R}^2 \setminus \overline{B(0,1)} = \{x \in \mathbb{R}^2 :\ |x| > 1\}.$$

Then by the variational principle,

$$N_-(\mathcal{E}_V) \leq N_-(\mathcal{E}_{V,1}) + N_-(\mathcal{E}_{V,2}) \tag{12}$$

(see, e.g., [10, Lemma 3.5]).
There exists an independent of V constant $C_1 > 0$ such that

$$N_-(\mathcal{E}_{V,1}) \leq C_1\|V\|_{\mathcal{B},\Omega_1} + 1 \tag{13}$$

(see [29, Theorems 4 and 4′, and Proposition 3]).
Below, we use the complex notation $z = x_1 + ix_2$ alongside the real one $x = (x_1, x_2)$. Let

$$\hat{V}(z) := \frac{1}{|z|^4} V\left(\frac{1}{z}\right), \qquad \tilde{w}(z) := w(1/z), \quad w \in W_2^1\left(\Omega_2\right), \qquad |z| < 1$$

(cf. the proof of Proposition 5.3 in [9]). An easy calculation gives

$$\begin{gathered}\int_{\Omega_2} |\nabla w(y)|^2 dy = \int_{\Omega_1} |\nabla \tilde{w}(x)|^2 dx, \\ \int_{\Omega_2} V(y)|w(y)|^2 dy = \int_{\Omega_1} \hat{V}(x)|\tilde{w}(x)|^2 dx,\end{gathered}$$

and it follows from (13) that

$$N_-(\mathcal{E}_{V,2}) \le C_1 \left\| \hat{V} \right\|_{\mathcal{B},\Omega_1} + 1. \tag{14}$$

Let us estimate the norm in the right-hand side. It is more convenient to work with the Luxemburg (gauge) norm (2). Using the notation $\zeta = y_1 + iy_2 = 1/z$, we get for any $\kappa > 0$

$$\begin{aligned}
\int_{\Omega_1} \mathcal{B}\left(\frac{\hat{V}(z)}{\kappa}\right) dx &= \int_{\Omega_1} \mathcal{B}\left(\frac{V(1/z)}{\kappa |z|^4}\right) dx = \int_{\Omega_2} \mathcal{B}\left(\frac{|\zeta|^4 V(\zeta)}{\kappa}\right) \frac{1}{|\zeta|^4}\, dy \\
&= \int_{\Omega_2} \left(\left(1 + \frac{1}{\kappa}\, |y|^4 V(y)\right) \ln\left(1 + \frac{1}{\kappa}\, |y|^4 V(y)\right) - \frac{1}{\kappa}\, |y|^4 V(y)\right) \frac{1}{|y|^4}\, dy \\
&\le \int_{\Omega_2} \left(\left(1 + \frac{1}{\kappa}\, |y|^4 V(y)\right) \ln\left(1 + \frac{1}{\kappa}\, V(y)\right) - \frac{1}{\kappa}\, |y|^4 V(y)\right) \frac{1}{|y|^4}\, dy \\
&\quad + \int_{\Omega_2} \frac{1}{|y|^4} \left(1 + \frac{1}{\kappa}\, |y|^4 V(y)\right) \ln\left(1 + |y|^4\right) dy \\
&\le \int_{\Omega_2} \left(\left(1 + \frac{1}{\kappa}\, V(y)\right) \ln\left(1 + \frac{1}{\kappa}\, V(y)\right) - \frac{1}{\kappa}\, V(y)\right) dy \\
&\quad + \frac{1}{\kappa} \int_{\Omega_2} V(y) \ln\left(1 + |y|^4\right) dy + \int_{\Omega_2} \frac{1}{|y|^4} \ln\left(1 + |y|^4\right) dy \\
&< \int_{\Omega_2} \mathcal{B}\left(\frac{V(y)}{\kappa}\right) dy + \frac{4}{\kappa} \int_{\Omega_2} V(y) \ln\left(1 + |y|\right) dy \\
&\quad + 4 \int_{\Omega_2} \frac{1}{|y|^4} \ln\left(1 + |y|\right) dy.
\end{aligned}$$

Let

$$\kappa_0 := \max\left\{ \|V\|_{(\mathcal{B},\Omega_2)},\ \int_{\Omega_2} V(y) \ln\left(1 + |y|\right) dy \right\}.$$

Then

$$\int_{\Omega_1} \mathcal{B}\left(\frac{\hat{V}(z)}{\kappa_0}\right) dx < 1 + 4 + 4 \int_{\Omega_2} |y|^{-3}\, dy = 5 + 8\pi =: C_2,$$

and

$$\left\| \hat{V} \right\|_{(\mathcal{B},\Omega_1)} \le C_2 \max\left\{ \|V\|_{(\mathcal{B},\Omega_2)},\ \int_{\Omega_2} V(y) \ln\left(1 + |y|\right) dy \right\}$$

(see (4)).
Now it follows from (12)–(14) that

$$N_-(\mathcal{E}_V) \le C_3 \left(\|V\|_{\mathcal{B},\mathbb{R}^2} + \int_{\mathbb{R}^2} V(x) \ln(1 + |x|)\, dx \right) + 2, \tag{15}$$

where one can take $C_3 = C_1(1+2C_2)$. The last inequality gives the estimate $N_-(\mathcal{E}_V) \le 2$ for small V and it is left to show that one actually has $N_-(\mathcal{E}_V) = 1$ in this case.
According to Proposition 4.4 in [9], $N_-(\mathcal{E}_V) = 1$ provided

$$\sup_{x'\in\mathbb{R}^2} \int_{\mathbb{R}^2} K(x,x')V(x)\,dx \tag{16}$$

is sufficiently small, where

$$\begin{aligned} &K(x,x') := \ln(2+|x|) + K_0(x,x'), \\ &K_0(x,x') := \ln_+ \frac{1}{|x-x'|}, \quad x, x' \in \mathbb{R}^2,\ x \ne x' \end{aligned}$$

(see (9)). It is easy to see that

$$\sup_{x'\in\mathbb{R}^2} \|K_0(\cdot,x')\|_{\mathcal{A},\mathbb{R}^2} = \|K_0(\cdot,0)\|_{\mathcal{A},\mathbb{R}^2} < \infty$$

(see (8)). Hence it follows from the Hölder inequality for Orlicz spaces (see [28, §3.3, (17) and (14)]) that (16) can be made arbitrarily small by making $\|V\|_{\mathcal{B},\mathbb{R}^2} + \|V\|_{L_1(\mathbb{R}^2,\,\ln(1+|x|)\,dx)}$ sufficiently small. In this case, $N_-(\mathcal{E}_V) = 1$. Combining this with (15) one gets the existence of a constant $C > 0$ for which (11) holds. $\square$

Remark 3.2. Estimate (11) looks similar to the following one obtained in [26]:

$$N_-(\mathcal{E}_V) \le C\left(\int_{V(x)\ge 1} V(x)\ln V(x)\,dx + \int_{\mathbb{R}^2} V(x)\ln(2+|x|)\,dx\right) + 1. \tag{17}$$

An advantage of (11) is that its right-hand side agrees with the semi-classical asymtotics $N_-(\mathcal{E}_{\alpha V}) = O(\alpha)$ as $\alpha \to +\infty$. In fact, Theorems 3.1 and 6.1 are direct descendants of (46) (see below) which was obtained in [29]. It turns out that the right-hand side of (11) dominates that of (46), i.e. that (46) is actually a stronger estimate than (11) (see Section 8).

4 The Khuri-Martin-Wu conjecture

Let $V_* : \mathbb{R}_+ \to [0,+\infty]$ be the non-increasing spherical rearrangement of V, i.e. let V_* be non-increasing, continuous from the right and such that

$$\left|\{x\in\mathbb{R}^2 : V_*(|x|) > s\}\right| = \left|\{x\in\mathbb{R}^2 : V(x) > s\}\right|, \quad \forall s > 0,$$

where $|E|$ denotes the two dimensional Lebesgue measure of $E \subset \mathbb{R}^2$. (Note that $V_*(r) = V^*(\pi r^2)$, where V^* is the standard non-increasing rearrangement used in the theory of Lorentz spaces; see, e.g., [33, 1.8].) Then

$$\int_{\mathbb{R}^2} F(V_*(|x|))dx = \int_{\mathbb{R}^2} F(V(x))dx \tag{18}$$

for any measurable $F \geq 0$ (see, e.g., [28, Ch. X, (1.10)]).

Lemma 4.1. (Cf. [2]) *There exists a constant $C_4 > 0$ such that*

$$\|V\|_{\mathcal{B},\mathbb{R}^2} \leq C_4 \left(\|V\|_{L_1(\mathbb{R}^2)} + \int_{\mathbb{R}^2} V_*(|x|) \ln_+ \frac{1}{|x|}\, dx \right), \quad \forall V \geq 0. \tag{19}$$

Proof. It follows from the devinition of V_* that

$$\begin{aligned} \left|\{x \in \mathbb{R}^2 : V(x) \geq V_*(r)\}\right| &= \left|\{x \in \mathbb{R}^2 : V_*(|x|) \geq V_*(r)\}\right| \\ &\geq \left|\{x \in \mathbb{R}^2 : |x| \leq r\}\right| = \pi r^2, \quad \forall r > 0. \end{aligned}$$

On the other hand, Chebyshev's inequality implies

$$\left|\{x \in \mathbb{R}^2 : V(x) \geq V_*(r)\}\right| \leq \frac{\|V\|_{L_1(\mathbb{R}^2)}}{V_*(r)}\, .$$

Hence

$$V_*(r) \leq \frac{\|V\|_{L_1(\mathbb{R}^2)}}{\pi r^2}\, . \tag{20}$$

Similarly to the proof of Theorem 3.1, it will be convenient for us to work with the norm (2). Let

$$\kappa_1 := \max\left\{ \frac{\|V\|_{L_1(\mathbb{R}^2)}}{\pi}\, ,\ \int_{\mathbb{R}^2} V_*(|x|) \ln_+ \frac{1}{|x|}\, dx \right\}.$$

Then it follows from Lemma 2.2 and (18), (20) that

$$\begin{aligned} &\int_{\mathbb{R}^2} \mathcal{B}\left(\frac{V(x)}{\kappa_1}\right) dx \leq \frac{1}{\kappa_1} \int_{\mathbb{R}^2} V(x)dx + \frac{2}{\kappa_1} \int_{\mathbb{R}^2} V(x) \ln_+ \frac{V(x)}{\kappa_1}\, dx \\ &\leq \pi + \frac{2}{\kappa_1} \int_{\mathbb{R}^2} V_*(|x|) \ln_+ \frac{V_*(|x|)}{\kappa_1}\, dx \leq \pi + \frac{2}{\kappa_1} \int_{\mathbb{R}^2} V_*(|x|) \ln_+ \frac{\|V\|_{L_1(\mathbb{R}^2)}}{\kappa_1 \pi |x|^2}\, dx \\ &\leq \pi + \frac{2}{\kappa_1} \int_{\mathbb{R}^2} V_*(|x|) \ln_+ \frac{1}{|x|^2}\, dx \leq \pi + 4 =: C_5. \end{aligned}$$

Hence

$$\|V\|_{(\mathcal{B},\mathbb{R}^2)} \leq C_5 \max\left\{ \frac{\|V\|_{L_1(\mathbb{R}^2)}}{\pi}\, ,\ \int_{\mathbb{R}^2} V_*(|x|) \ln_+ \frac{1}{|x|}\, dx \right\}$$

(see (4)), which implies (19) with $C_4 = 2C_5$ (see (3)). ∎

Lemma 4.2. (Cf. [2])

$$\int_{\mathbb{R}^2} V_*(|x|) \ln_+ \frac{1}{|x|}\, dx \le 4\pi \|V\|_{\mathcal{B},\mathbb{R}^2}, \quad \forall V \ge 0.$$

Proof. Let $\mathbb{D} := \{x \in \mathbb{R}^2 : \ |x| \le 1\}$. The Hölder inequality (see [28, §3.3, (4)]) and (18) imply

$$\int_{\mathbb{R}^2} V_*(|x|) \ln_+ \frac{1}{|x|}\, dx = \int_{\mathbb{D}} V_*(|x|) \ln \frac{1}{|x|}\, dx \le 2\|V_*(|\cdot|)\|_{(\mathcal{B},\mathbb{D})} \left\| \ln \frac{1}{|\cdot|} \right\|_{(\mathcal{A},\mathbb{D})}$$
$$\le 2\|V_*(|\cdot|)\|_{(\mathcal{B},\mathbb{R}^2)} \left\| \ln \frac{1}{|\cdot|} \right\|_{(\mathcal{A},\mathbb{D})} = 2\|V\|_{(\mathcal{B},\mathbb{R}^2)} \left\| \ln \frac{1}{|\cdot|} \right\|_{(\mathcal{A},\mathbb{D})}.$$

$$\int_{\mathbb{D}} \mathcal{A}\left(\ln \frac{1}{|x|}\right) dx \le \int_{\mathbb{D}} e^{\ln \frac{1}{|x|}}\, dx = \int_{\mathbb{D}} \frac{1}{|x|}\, dx = \int_{-\pi}^{\pi} \int_0^1 1\, dr d\vartheta = 2\pi.$$

Hence

$$\left\| \ln \frac{1}{|\cdot|} \right\|_{(\mathcal{A},\mathbb{D})} \le 2\pi$$

(see (4)). □

Lemmas 4.1 and 4.2 imply that Theorem 3.1 is equivalent to the following result.

Theorem 4.3. *There exists a constant $C_6 > 0$ such that*

$$N_-(\mathcal{E}_V) \le C_6 \left(\int_{\mathbb{R}^2} V(x) \ln(2 + |x|)\, dx + \int_{\mathbb{R}^2} V_*(|x|) \ln_+ \frac{1}{|x|}\, dx \right) + 1, \quad \forall V \ge 0. \tag{21}$$

Theorem 4.3 is in turn equivalent to the following estimate which was conjectured in [17]

$$\begin{aligned} N_-(\mathcal{E}_V) \ &\le \ c_1 \int_{\mathbb{R}^2} V_*(|x|) \ln_+ \frac{1}{|x|}\, dx + c_2 \int_{\mathbb{R}^2} V(x) \ln_+ |x|\, dx \\ &+ \ c_3 \int_{\mathbb{R}^2} V(x)\, dx + 1, \end{aligned} \tag{22}$$

provided one does not care about the exact values of the constants. It is natural however to ask what the best constants in (22) are. It was conjectured in [7] that a similar estimate

$$\begin{aligned} N_-(\mathcal{E}_V) \ &\le \ d_1 \int_{\mathbb{R}^2} V_*(|x|) \ln_+ \frac{1}{|x|}\, dx + d_2 \int_{\mathbb{R}^2} V(x) \ln |x|\, dx \\ &+ \ d_3 \int_{\mathbb{R}^2} V(x)\, dx + 1 \end{aligned} \tag{23}$$

holds with

$$2\pi d_1 = 2, \quad 2\pi d_2 = 1, \quad 2\pi d_3 = \frac{2}{\sqrt{3}}\,. \tag{24}$$

The difference between (22) and (23) is that the second integral in the latter can be negative. If $V(x) = F(|x|)$ is a decreasing radial potential, then (22) and (23) become

$$\begin{aligned} N_-(\mathcal{E}_V) &\le 2\pi c_1 \int_0^1 rF(r)|\ln r|\,dr + 2\pi c_2 \int_1^\infty rF(r)\ln r\,dr \\ &+ 2\pi c_3 \int_0^\infty rF(r)\,dr + 1 \end{aligned}$$

and

$$\begin{aligned} N_-(\mathcal{E}_V) &\le 2\pi(d_1 - d_2) \int_0^1 rF(r)|\ln r|\,dr + 2\pi d_2 \int_1^\infty rF(r)\ln r\,dr \\ &+ 2\pi d_3 \int_0^\infty rF(r)\,dr + 1 \end{aligned}$$

respectively. This allows one to obtain lower estimates for c_1, c_2, c_3 and d_1, d_2, d_3 from the known results on radial potentials and explains the values of d_1 and d_2 in (24). Theorem 5.2 in [20] suggests that one could perhaps try to prove (23) with $2\pi d_3 = 1$.

Remark 4.4. The proof of Theorem 4.3 relies on an idea from [26, Section 5] where (17) was used in place of (11).

5 The Birman-Solomyak method

Our description of the Birman-Solomyak method of estimating $N_-(\mathcal{E}_V)$ follows [4, 29, 31], although some details are different.
We denote the polar coordinates in $\mathbb{R}^2$ by (r, ϑ), $r \in \mathbb{R}_+$, $\vartheta \in= (-\pi, \pi]$. Let

$$f_{\mathcal{R}}(r) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(r,\vartheta)\,d\vartheta, \quad f_{\mathcal{N}}(r,\vartheta) := f(r,\vartheta) - f_{\mathcal{R}}(r), \quad f \in C\left(\mathbb{R}^2 \setminus \{0\}\right).$$

Then

$$\int_{-\pi}^{\pi} f_{\mathcal{N}}(r,\vartheta)\,d\vartheta = 0, \quad \forall r > 0, \tag{25}$$

and it is easy to see that

$$\int_{\mathbb{R}^2} f_{\mathcal{R}} g_{\mathcal{N}}\,dx = 0, \quad \forall f, g \in C\left(\mathbb{R}^2 \setminus \{0\}\right) \cap L_2\left(\mathbb{R}^2\right).$$

Hence $f \mapsto Pf := f_{\mathcal{R}}$ extends to an orthogonal projection $P : L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$.
Using the representation of the gradient in polar coordinates one gets

$$\int_{\mathbb{R}^2} \nabla f_{\mathcal{R}} \nabla g_{\mathcal{N}}\, dx = \int_{\mathbb{R}^2} \left(\frac{\partial f_{\mathcal{R}}}{\partial r} \frac{\partial g_{\mathcal{N}}}{\partial r} + \frac{1}{r^2} \frac{\partial f_{\mathcal{R}}}{\partial \vartheta} \frac{\partial g_{\mathcal{N}}}{\partial \vartheta} \right) dx$$
$$= \int_{\mathbb{R}^2} \frac{\partial f_{\mathcal{R}}}{\partial r} \frac{\partial g_{\mathcal{N}}}{\partial r}\, dx = \int_{\mathbb{R}^2} \left(\frac{\partial f}{\partial r} \right)_{\mathcal{R}} \left(\frac{\partial g}{\partial r} \right)_{\mathcal{N}} dx = 0, \quad \forall f, g \in C_0^\infty\left(\mathbb{R}^2\right).$$

Hence $P : W_2^1(\mathbb{R}^2) \to W_2^1(\mathbb{R}^2)$ is also an orthogonal projection.
Since

$$\int_{\mathbb{R}^2} |\nabla w|^2\, dx = \int_{\mathbb{R}^2} |\nabla w_{\mathcal{R}}|^2\, dx + \int_{\mathbb{R}^2} |\nabla w_{\mathcal{N}}|^2\, dx,$$
$$\int_{\mathbb{R}^2} V|w|^2\, dx \le 2 \int_{\mathbb{R}^2} V|w_{\mathcal{R}}|^2\, dx + 2 \int_{\mathbb{R}^2} V|w_{\mathcal{N}}|^2\, dx,$$

one has

$$N_-(\mathcal{E}_V) \le N_-(\mathcal{E}_{\mathcal{R},2V}) + N_-(\mathcal{E}_{\mathcal{N},2V}), \tag{26}$$

where $\mathcal{E}_{\mathcal{R},2V}$ and $\mathcal{E}_{\mathcal{N},2V}$ denote the restrictions of the form $\mathcal{E}_{2V}$ to $PW_2^1(\mathbb{R}^2)$ and $(I-P)W_2^1(\mathbb{R}^2)$ respectively.

Remark 5.1. If the potential V is radial, i.e. $V(x) = F(r)$, then it is easy to see that

$$P(Vw) = VPw, \quad \forall w \in L_2\left(\mathbb{R}^2\right) \cap L_2\left(\mathbb{R}^2, V(x)dx\right),$$

and one gets a sharper version of (26):

$$N_-(\mathcal{E}_V) = N_-(\mathcal{E}_{\mathcal{R},V}) + N_-(\mathcal{E}_{\mathcal{N},V}) \tag{27}$$

(cf. [20]).

Let us estimate the right-hand side of (26). We start with the first term, i.e with the case of $PW_2^1(\mathbb{R}^2) = \{w \in W_2^1(\mathbb{R}^2) : \ w(x) = w_{\mathcal{R}}(r)\}$. Using the notation $r = e^t$, $w(x) = w_{\mathcal{R}}(r) = u(t)$, we get

$$\int_{\mathbb{R}^2} |\nabla w(x)|^2\, dx = 2\pi \int_{\mathbb{R}} |u'(t)|^2\, dt, \quad \int_{\mathbb{R}^2} |w(x)|^2\, dx = 2\pi \int_{\mathbb{R}} |u(t)|^2 e^{2t}\, dt,$$
$$\int_{\mathbb{R}^2} V(x)|w(x)|^2\, dx = 2\pi \int_{\mathbb{R}} G(t)|u(t)|^2\, dt,$$

where

$$G(t) := \frac{e^{2t}}{2\pi} \int_{-\pi}^{\pi} V(e^t, \vartheta)\, d\vartheta. \tag{28}$$

The above change of variable defines a unitary operator from $PW_2^1(\mathbb{R}^2)$ onto

$$X := \left\{ u \in W^1_{2,\mathrm{loc}}(\mathbb{R}) : \ \|u\|_X := \sqrt{2\pi} \left(\int_{\mathbb{R}} |u'|^2 \, dt + \int_{\mathbb{R}} |u|^2 e^{2t} \, dt \right)^{1/2} < \infty \right\}.$$

Let $X_0 := \{u \in X : \ u(0) = 0\}$ and

$$\mathcal{H}_0 := \left\{ u \in W^1_{2,\mathrm{loc}}(\mathbb{R}) : \ u(0) = 0, \ \int_{\mathbb{R}} |u'|^2 \, dt < \infty \right\}. \tag{29}$$

Let $\mathcal{E}_{X,G}$, $\mathcal{E}_{X_0,G}$ and $\mathcal{E}_{\mathcal{H}_0,G}$ denote the forms defined by

$$\int_{\mathbb{R}} |u'(t)|^2 \, dt - \int_{\mathbb{R}} G(t) |u(t)|^2 \, dt \tag{30}$$

on the domains

$$X \cap L_2(\mathbb{R}, G(t)dt), \quad X_0 \cap L_2(\mathbb{R}, G(t)dt) \quad \text{and} \quad \mathcal{H}_0 \cap L_2(\mathbb{R}, G(t)dt)$$

respectively. Since $\dim(X/X_0) = 1$ and $X_0 \subset \mathcal{H}_0$, one has

$$N_-(\mathcal{E}_{\mathcal{R},2V}) = N_-(\mathcal{E}_{X,2G}) \le N_-(\mathcal{E}_{X_0,2G}) + 1 \le N_-(\mathcal{E}_{\mathcal{H}_0,2G}) + 1.$$

It follows from Hardy's inequality (see, e.g., [14, Theorem 327]) that

$$\int_{\mathbb{R}} |u'|^2 \, dt + \kappa \int_{\mathbb{R}} \frac{|u|^2}{|t|^2} \, dt \le (4\kappa + 1) \int_{\mathbb{R}} |u'|^2 \, dt, \quad \forall u \in \mathcal{H}_0, \quad \forall \kappa \ge 0.$$

Hence

$$N_-(\mathcal{E}_{\mathcal{H}_0,2G}) \le N_-(\mathcal{E}_{\kappa,G}),$$

where

$$\mathcal{E}_{\kappa,G}[u] := \int_{\mathbb{R}} |u'(t)|^2 \, dt + \kappa \int_{\mathbb{R}} \frac{|u(t)|^2}{|t|^2} \, dt - 2(4\kappa + 1) \int_{\mathbb{R}} G(t) |u(t)|^2 \, dt, \tag{31}$$
$$\mathrm{Dom}\,(\mathcal{E}_{\kappa,G}) = \mathcal{H}_0 \cap L_2(\mathbb{R}, G(t)dt).$$

It follows from the above that

$$N_-(\mathcal{E}_{\mathcal{R},2V}) \le N_-(\mathcal{E}_{\kappa,G}) + 1. \tag{32}$$

We estimate $N_-(\mathcal{E}_{\kappa,G})$ by partitioning $\mathbb{R}$ into the intervals

$$I_n := [2^{n-1}, 2^n], \ n > 0, \quad I_0 := [-1, 1], \quad I_n := [-2^{|n|}, -2^{|n|-1}], \ n < 0,$$

and by using the variational principle to obtain

$$N_-(\mathcal{E}_{\kappa,G}) \le \sum_{n\in\mathbb{Z}} N_-(\mathcal{E}_{\kappa,G,n}), \tag{33}$$

where

$$\begin{aligned}
&\mathcal{E}_{\kappa,G,n}[u] := \int_{I_n} |u'|^2\,dt + \kappa \int_{I_n} \frac{|u|^2}{|t|^2}\,dt - 2(4\kappa+1)\int_{I_n} G|u|^2\,dt,\\
&\mathrm{Dom}\,(\mathcal{E}_{\kappa,G,n}) = W_2^1(I_n) \cap L_2\left(I_n, G(t)dt\right),\ n \in \mathbb{Z}\setminus\{0\},\\
&\mathrm{Dom}\,(\mathcal{E}_{\kappa,G,0}) = \{u \in W_2^1(I_0):\ u(0) = 0\} \cap L_2\left(I_0, G(t)dt\right).
\end{aligned}$$

Let $n > 0$. For any $N \in \mathbb{N}$, there exists a subspace $\mathcal{F}_N \in \mathrm{Dom}\,(\mathcal{E}_{\kappa,G,n})$ of co-dimension N such that

$$\int_{I_n} G|u|^2\,dt \le \left(\frac{|I_n|}{N^2}\int_{I_n} G\,dt\right)\int_{I_n} |u'|^2\,dt, \quad \forall u \in \mathcal{F}_N$$

(see [31, the proof of Proposition 4.2 in Appendix]). If

$$2(4\kappa+1)\,\frac{|I_n|}{N^2}\int_{I_n} G\,dt \le 1,$$

then $\mathcal{E}_{\kappa,G,n}[u] \ge 0$, $\forall u \in \mathcal{F}_N$, and $N_-(\mathcal{E}_{\kappa,G,n}) \le N$. Let

$$\mathcal{A}_n := \int_{I_n} |t|G(t)\,dt,\ n \ne 0, \quad \mathcal{A}_0 := \int_{I_0} G(t)\,dt. \tag{34}$$

Since $|I_n| \int_{I_n} G\,dt \le \mathcal{A}_n$, $n \ne 0$, it follows from the above that

$$2(4\kappa+1)\mathcal{A}_n \le N^2 \implies N_-(\mathcal{E}_{\kappa,G,n}) \le N.$$

Hence

$$N_-(\mathcal{E}_{\kappa,G,n}) \le \left\lceil \sqrt{2(4\kappa+1)\mathcal{A}_n}\right\rceil, \tag{35}$$

where $\lceil\cdot\rceil$ denotes the ceiling function, i.e. $\lceil a\rceil$ is the smallest integer not less than a. The right-hand side of this estimate is at least 1, so one cannot feed it straight into (33). One needs to find conditions under which $N_-(\mathcal{E}_{\kappa,G,n}) = 0$. It follows from (79), (81) that

$$\int_{I_n} G|u|^2\,dt \le \mathcal{A}_n C(\kappa)\left(\int_{I_n} |u'|^2\,dt + \kappa\int_{I_n} \frac{|u|^2}{|t|^2}\,dt\right),$$

where

$$C(\kappa) = \frac{1}{2\kappa}\left(1 + \sqrt{1+4\kappa}\,\frac{2^{\sqrt{1+4\kappa}}+1}{2^{\sqrt{1+4\kappa}}-1}\right).$$

Hence $N_-(\mathcal{E}_{\kappa,G,n}) = 0$, i.e. $\mathcal{E}_{\kappa,G,n} \ge 0$, provided $\mathcal{A}_n \le \Phi(\kappa)$, where

$$\Phi(\kappa) := \frac{\kappa}{4\kappa+1}\left(1 + \sqrt{4\kappa+1}\,\frac{2^{\sqrt{4\kappa+1}}+1}{2^{\sqrt{4\kappa+1}}-1}\right)^{-1}. \tag{36}$$

The above estimates for $N_-(\mathcal{E}_{\kappa,G,n})$ clearly hold for $n<0$ as well, but the case $n=0$ requires some changes. Since $u(0)=0$ for any $u \in \operatorname{Dom}(\mathcal{E}_{\kappa,G,0})$, one can use the same argument as the one leading to (35), but with two differences: a) $\mathcal{F}_N$ can be chosen to be of co-dimension $N-1$, and b) $|I_0|\int_{I_0} G\,dt = 2\mathcal{A}_0$. This gives the following analogue of (35)

$$N_-(\mathcal{E}_{\kappa,G,0}) \le \left\lceil 2\sqrt{(4\kappa+1)\mathcal{A}_0}\right\rceil - 1 < 2\sqrt{(4\kappa+1)\mathcal{A}_0}\,.$$

In particular, $N_-(\mathcal{E}_{\kappa,G,0}) = 0$ if $\mathcal{A}_0 \le 1/(4(4\kappa+1))$. Using Remark 10.2, one can easily see that the implication $\mathcal{A}_n \le \Phi(\kappa) \implies N_-(\mathcal{E}_{\kappa,G,n}) = 0$ remains true for $n=0$. Now it follows from (32), (33) that

$$N_-(\mathcal{E}_{\mathcal{R},2V}) \le 1 + \sum_{\{n\in\mathbb{Z}\setminus\{0\}:\ \mathcal{A}_n > \Phi(\kappa)\}} \left\lceil \sqrt{2(4\kappa+1)\mathcal{A}_n}\right\rceil + 2\sqrt{(4\kappa+1)\mathcal{A}_0}\,, \tag{37}$$

and one can drop the last term in $\mathcal{A}_0 \le \Phi(\kappa)$. The presence of the parameter κ in this estimate allows a degree of flexibility. In order to decrease the number of terms in the sum in the right-hand side, one should choose κ in such a way that $\Phi(\kappa)$ is close to its maximum. A Mathematica calculation shows that the maximum is approximately 0.046 and is achieved at $\kappa \approx 1.559$. For values of κ close to 1.559, one has

$$\mathcal{A}_n > \Phi(\kappa) \implies \sqrt{2(4\kappa+1)\mathcal{A}_n} > \sqrt{2(4\kappa+1)\Phi(\kappa)} \approx 0.816.$$

Since $\lceil a\rceil \le 2a$ for $a \ge 1/2$, (37) implies

$$N_-(\mathcal{E}_{\mathcal{R},2V}) \le 1 + 2\sqrt{2(4\kappa+1)} \sum_{\mathcal{A}_n > \Phi(\kappa)} \sqrt{\mathcal{A}_n}$$

with $\kappa \approx 1.559$. Hence

$$N_-(\mathcal{E}_{\mathcal{R},2V}) \le 1 + 7.61 \sum_{\mathcal{A}_n > 0.046} \sqrt{\mathcal{A}_n}\,.$$

Let us rewrite this estimate in terms of the original potential V. Set

$$\begin{aligned} U_n &:= \{x \in \mathbb{R}^2 :\ e^{2^{n-1}} < |x| < e^{2^n}\},\ n>0,\\ U_0 &:= \{x \in \mathbb{R}^2 :\ e^{-1} < |x| < e\},\\ U_n &:= \{x \in \mathbb{R}^2 :\ e^{-2^{|n|}} < |x| < e^{-2^{|n|-1}}\},\ n<0, \end{aligned} \tag{38}$$

and

$$A_0 := \int_{U_0} V(x)\,dx, \quad A_n := \int_{U_n} V(x)|\ln|x||\,dx, \ n \neq 0. \tag{39}$$

Then it follows from (28), (34) that $A_n = 2\pi\mathcal{A}_n$, and we have

$$N_-(\mathcal{E}_{\mathcal{R},2V}) \le 1 + 3.04 \sum_{A_n > 0.29} \sqrt{A_n}\,.$$

Below, we will use the following less precise but nicer looking estimate

$$N_-(\mathcal{E}_{\mathcal{R},2V}) \le 1 + 4 \sum_{A_n > 1/4} \sqrt{A_n}\,. \tag{40}$$

Remark 5.2. To the best of my knowledge, estimates of this type (without explicit constants) were first obtained by M. Birman and M. Solomyak for Schrödinger-type operators of order 2ℓ in $\mathbb{R}^d$ with $2\ell > d$ (see [6, §6]). A. Grigor'yan and N. Nadirashvili ([10]) obtained an estimate of this type for two-dimensional Schrödinger operators (see (49) below). The Grigor'yan-Nadirashvili estimate is discusses in the next section.

Returning to (26), we need to estimate $N_-(\mathcal{E}_{\mathcal{N},2V})$. To this end, we split $\mathbb{R}^2$ into the following unnuli

$$\Omega_n := \{x \in \mathbb{R}^2 : \ e^n < |x| < e^{n+1}\}, \quad n \in \mathbb{Z}. \tag{41}$$

It follows from (25) that

$$\int_{\Omega_n} w(x)\,dx = 0, \quad \forall w \in (I - P)W_2^1\left(\mathbb{R}^2\right), \ \ \forall n \in \mathbb{Z}.$$

Hence the variational principle implies

$$N_-(\mathcal{E}_{\mathcal{N},2V}) \le \sum_{n\in\mathbb{Z}} N_-(\mathcal{E}_{\mathcal{N},2V,n}), \tag{42}$$

where

$$\begin{gathered}\mathcal{E}_{\mathcal{N},2V,n}[w] := \textstyle\int_{\Omega_n} |\nabla w(x)|^2 dx - 2\int_{\Omega_n} V(x)|w(x)|^2 dx, \\ \operatorname{Dom}(\mathcal{E}_{\mathcal{N},2V,n}) = \left\{w \in W_2^1(\Omega_n) \cap L_2(\Omega_n, V(x)dx) : \ \textstyle\int_{\Omega_n} w\,dx = 0\right\}.\end{gathered} \tag{43}$$

It is clear that $x \mapsto e^n x$ maps U_0 onto U_n. So, any estimate for $N_-(\mathcal{E}_{\mathcal{N},2V,0})$ that has the right scaling leads to an estimate for $N_-(\mathcal{E}_{\mathcal{N},2V,n})$, and then an estimate for $N_-(\mathcal{E}_V)$ follows from (26), (40) and (42).

6 A Grigor'yan-Nadirashvili type estimate

Let

$$\mathcal{B}_n := \|V\|^{(\mathrm{av})}_{\mathcal{B},\Omega_n}, \quad B_n := \left(\int_{\Omega_n} V^p(x)|x|^{2(p-1)}\,dx\right)^{1/p}, \quad p > 1, \quad n \in \mathbb{Z} \tag{44}$$

(see (41)).

Theorem 6.1. *There exist constants $C_7 > 0$ and $c > 0$ such that*

$$N_-(\mathcal{E}_V) \le 1 + 4 \sum_{A_n > 1/4} \sqrt{A_n} + C_7 \sum_{\mathcal{B}_n > c} \mathcal{B}_n, \quad \forall V \ge 0 \tag{45}$$

(see (38), (39)*).*

Proof. According to Theorem 4′ in [29] (see also Proposition 3 there), there exists $C_7 > 0$ such that $N_-(\mathcal{E}_{\mathcal{N},2V,n}) \le C_7\mathcal{B}_n$, $\forall n \in \mathbb{Z}$ (see (43)). In particular, $N_-(\mathcal{E}_{\mathcal{N},2V,n}) = 0$ if $\mathcal{B}_n < 1/C_7$, and one can drop this term from the sum in (42). Now it follows from (26), (40) and (42) that (45) holds for any $c < 1/C_7$. □

Remark 6.2. The above result is closely related to the well known estimate from [29]:

$$N_-(\mathcal{E}_V) \le 1 + C_8 \left(\|(\mathbf{A}_n)_{n\ge 0}\|_{1,\infty} + \sum_{n=0}^{\infty} \mathbf{B}_n\right), \quad \forall V \ge 0, \tag{46}$$

where $\mathbf{A}_n = A_n$, $\mathbf{B}_n = \mathcal{B}_n$ for $n \in \mathbb{N}$,

$$\mathbf{A}_0 := \int_{\mathbf{\Omega}_0} V(x)|\ln|x||\,dx, \quad \mathbf{B}_0 = \|V\|^{(\mathrm{av})}_{\mathcal{B},\mathbf{\Omega}_0}, \quad \mathbf{\Omega}_0 = \{x \in \mathbb{R}^2 : \ |x| \le e\}, \tag{47}$$

and

$$\|(a_n)_{n\ge 0}\|_{1,\infty} := \sup_{s>0} \Big(s\,\mathrm{card}\,\{n \ge 0 : \ |a_n| > s\}\Big).$$

Note that

$$\sum_{\{n\le 0:\ \mathcal{B}_n > c\}} \mathcal{B}_n \le \sum_{n\le 0} \mathcal{B}_n \le \mathbf{B}_0$$

(see [29, Lemma 3]), and that

$$\begin{aligned}
&\sum_{|a_n|>c} \sqrt{|a_n|} = \sum_{|a_n|>c} \int_0^{|a_n|} \frac{1}{2}\, s^{-1/2} ds \\
&= \frac{1}{2} \int_0^\infty s^{-1/2}\, \mathrm{card}\,\{n \ge 0 :\ |a_n| > s \ \&\ |a_n| > c\}\, ds \\
&\le \frac{1}{2} \int_0^\infty s^{-1/2}\, \frac{\|(a_n)_{n\ge 0}\|_{1,\infty}}{\max\{s, c\}}\, ds \qquad (48)\\
&= \frac{\|(a_n)_{n\ge 0}\|_{1,\infty}}{2} \left(\int_0^c \frac{s^{-1/2}}{c}\, ds + \int_c^\infty s^{-3/2} ds \right) = \frac{2}{\sqrt{c}}\, \|(a_n)_{n\ge 0}\|_{1,\infty}\, .
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{\{n\in\mathbb{Z}\setminus\{0\}:\ A_n>c\}} \sqrt{A_n} = \sum_{\{n<0:\ A_n>c\}} \sqrt{A_n} + \sum_{\{n>0:\ A_n>c\}} \sqrt{A_n} \\
&\le \frac{1}{\sqrt{c}} \sum_{n<0} A_n + \frac{2}{\sqrt{c}}\, \|(A_n)_{n>0}\|_{1,\infty} \le \frac{1}{\sqrt{c}}\, \mathbf{A}_0 + \frac{2}{\sqrt{c}}\, \|(A_n)_{n>0}\|_{1,\infty} \\
&\le \frac{3}{\sqrt{c}}\, \|(\mathbf{A}_n)_{n\ge 0}\|_{1,\infty}\, .
\end{aligned}$$

It follows from the Hölder inequality (see [18, Theorem 9.3]) that if $A_0 > c$, then

$$\begin{aligned}
&\sqrt{A_0} < \frac{1}{\sqrt{c}}\, A_0 = \frac{1}{\sqrt{c}} \int_{U_0} V(x)\, dx \le \frac{1}{\sqrt{c}} \int_{\mathbf{\Omega}_0} V(x)\, dx \\
&\le \frac{1}{\sqrt{c}} \|V\|_{\mathcal{B},\mathbf{\Omega}_0} \|1\|_{\mathcal{A},\mathbf{\Omega}_0} \le \mathrm{const}\, \|V\|^{(\mathrm{av})}_{\mathcal{B},\mathbf{\Omega}_0} = \mathrm{const}\, \mathbf{B}_0.
\end{aligned}$$

So, (45) implies (46).

Remark 6.3. Theorem 6.1 is a slight improvement of Theorem 1.1 in [10] where an analogue of (45) was obtained with B_n in place of $\mathcal{B}_n$:

$$N_-(\mathcal{E}_V) \le 1 + C_7 \sum_{\{n\in\mathbb{Z}:\ A_n>c\}} \sqrt{A_n} + C_7 \sum_{\{n\in\mathbb{Z}:\ B_n>c\}} B_n, \quad \forall V \ge 0. \qquad (49)$$

Indeed, let

$$\begin{aligned}
&\Delta := \left\{ x \in \mathbb{R}^2 :\ \frac{1}{\sqrt{\pi(e^2-1)}} < |x| < \frac{e}{\sqrt{\pi(e^2-1)}} \right\}, \\
&R_n := e^n \sqrt{\pi(e^2-1)}\, , \\
&\xi_n : \mathbb{R}^2 \to \mathbb{R}^2, \quad \xi_n(x) = R_n x, \quad n \in \mathbb{Z}.
\end{aligned}$$

Then $|\Delta| = 1$, $\xi_n(\Delta) = \Omega_n$ and it follows from [29, (7)] and [18, §13] that

$$\begin{aligned}
&\mathcal{B}_n = \|V\|^{(\mathrm{av})}_{\mathcal{B},\Omega_n} = |\Omega_n| \|V \circ \xi_n\|_{\mathcal{B},\Delta} \le C(p)|\Omega_n| \|V \circ \xi_n\|_{L_p(\Delta)} \\
&= C(p) R_n^2 R_n^{-2/p} \|V\|_{L_p(\Omega_n)} = C(p) R_n^{2-2/p} \left(\int_{\Omega_n} V^p(x)dx\right)^{1/p} \\
&\le C(p) \left(\pi(e^2-1)\right)^{1-1/p} \left(\int_{\{e^n<|x|<e^{n+1}\}} V^p(x)|x|^{2(p-1)}\, dx\right)^{1/p} \\
&= C(p) \left(\pi(e^2-1)\right)^{1-1/p} B_n, \quad p > 1.
\end{aligned}$$

7 A Laptev-Netrusov-Solomyak type estimate

We denote the polar coordinates in $\mathbb{R}^2$ by (r, ϑ), $r \in \mathbb{R}_+$, $\vartheta \in \mathbb{S} := (-\pi, \pi]$. Let $I \subseteq \mathbb{R}_+$ be a nonempty open interval and let

$$\Omega_I := \{x \in \mathbb{R}^2 : \ |x| \in I\}.$$

We denote by $\mathcal{L}_1(I, L_{\mathcal{B}}(\mathbb{S}))$ the space of measurable functions $f : \Omega_I \to \mathbb{C}$ such that

$$\|f\|_{\mathcal{L}_1(I,L_{\mathcal{B}}(\mathbb{S}))} := \int_I \|f(r,\cdot)\|_{\mathcal{B},\mathbb{S}}\, r dr < +\infty. \tag{50}$$

Let

$$\mathcal{I}_n := (e^n, e^{n+1}), \quad \mathcal{D}_n := \|V\|_{\mathcal{L}_1(\mathcal{I}_n, L_{\mathcal{B}}(\mathbb{S}))}, \quad n \in \mathbb{Z}. \tag{51}$$

Theorem 7.1. *There exist constants $C_9 > 0$ and $c > 0$ such that*

$$N_-(\mathcal{E}_V) \le 1 + 4 \sum_{A_n > 1/4} \sqrt{A_n} + C_9 \sum_{\mathcal{D}_n > c} \mathcal{D}_n\,, \quad \forall V \ge 0 \tag{52}$$

(see (38), (39)*).*

Remark 7.2. It is clear that

$$\sum_{\{n\in\mathbb{Z}:\ \mathcal{D}_n > c\}} \mathcal{D}_n \le \sum_{n\in\mathbb{Z}} \mathcal{D}_n = \|V\|_{\mathcal{L}_1(\mathbb{R}_+, L_{\mathcal{B}}(\mathbb{S}))}.$$

It follows from (48) that

$$\sum_{\{n\in\mathbb{Z}:\ A_n > c\}} \sqrt{A_n} \le \frac{2}{\sqrt{c}}\, \|(A_n)_{n\in\mathbb{Z}}\|_{1,\infty}\,.$$

Hence (52) implies

$$N_-(\mathcal{E}_V) \le 1 + C_{10} \left(\|(A_n)_{n\in\mathbb{Z}}\|_{1,\infty} + \|V\|_{\mathcal{L}_1(\mathbb{R}_+, L_{\mathcal{B}}(\mathbb{S}))}\right), \quad \forall V \ge 0, \tag{53}$$

which, in turn, implies the following estimate obtained in [21] (see also [19]):

$$N_-(\mathcal{E}_V) \le 1 + C_{11} \left(\|(A_n)_{n\in\mathbb{Z}}\|_{1,\infty} + \int_{\mathbb{R}_+} \left(\int_{\mathbb{S}} |V(r,\vartheta)|^p d\vartheta \right)^{1/p} r dr \right), \quad (54)$$

where $p > 1$. In fact, the estimate obtained in [21] has a slightly different form:

$$N_-(\mathcal{E}_V) \le 1 + C_{12} \left(\|(A_n)_{n\in\mathbb{Z}}\|_{1,\infty} + \int_{\mathbb{R}_+} \left(\int_{\mathbb{S}} |V_{\mathcal{N}}(r,\vartheta)|^p d\vartheta \right)^{1/p} r dr \right), \quad (55)$$

where

$$V_{\mathcal{N}}(r,\vartheta) := V(r,\vartheta) - V_{\mathcal{R}}(r), \quad V_{\mathcal{R}}(r) := \frac{1}{2\pi} \int_{\mathbb{S}} V(r,\vartheta) d\vartheta.$$

Since

$$\begin{aligned}
&\left| \int_{\mathbb{R}_+} \left(\int_{\mathbb{S}} |V(r,\vartheta)|^p d\vartheta \right)^{1/p} r dr - \int_{\mathbb{R}_+} \left(\int_{\mathbb{S}} |V_{\mathcal{N}}(r,\vartheta)|^p d\vartheta \right)^{1/p} r dr \right| \\
&\quad \le \int_{\mathbb{R}_+} \left(\int_{\mathbb{S}} |V_{\mathcal{R}}(r)|^p d\vartheta \right)^{1/p} r dr = (2\pi)^{1/p} \int_{\mathbb{R}_+} V_{\mathcal{R}}(r)\, r dr \\
&\quad = (2\pi)^{1/p-1} \int_{\mathbb{R}^2} V(x)\, dx = (2\pi)^{1/p-1} \sum_{n\in\mathbb{Z}} \int_{U_n} V(x)\, dx \\
&\quad \le \text{const} \sum_{n\in\mathbb{Z}} 2^{-|n|} A_n \le \text{const} \sup_{n\in\mathbb{Z}} A_n \le \text{const}\ \|(A_n)_{n\in\mathbb{Z}}\|_{1,\infty},
\end{aligned}$$

estimates (54) and (55) are equivalent to each other. An advantage of the latter is that it separates the contribution of the radial part $V_{\mathcal{R}}$ of V from that of the non-radial part $V_{\mathcal{N}}$ (see [21]). Similarly to the above, one can easily show that (53) is equivalent to

$$N_-(\mathcal{E}_V) \le 1 + C_{13} \left(\|(A_n)_{n\in\mathbb{Z}}\|_{1,\infty} + \|V_{\mathcal{N}}\|_{\mathcal{L}_1(\mathbb{R}_+, L_{\mathcal{B}}(\mathbb{S}))} \right), \quad \forall V \ge 0. \quad (56)$$

Let $I_1, I_2 \subseteq \mathbb{R}$ be nonempty open intervals. We denote by $L_1(I_1, L_{\mathcal{B}}(I_2))$ the space of measurable functions $f : I_1 \times I_2 \to \mathbb{C}$ such that

$$\|f\|_{L_1(I_1, L_{\mathcal{B}}(I_2))} := \int_{I_1} \|f(x_1, \cdot)\|^{(\mathrm{av})}_{\mathcal{B}, I_2}\, dx_1 < +\infty \quad (57)$$

(see (7)).

Lemma 7.3. (Cf. [29, Lemma 1]) *Consider an affine transformation*

$$\xi : \mathbb{R}^2 \to \mathbb{R}^2,\ \xi(x) := Ax + x^0, \quad A = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix},\ R_1, R_2 > 0, \quad x^0 \in \mathbb{R}^2.$$

Let $I_1 \times I_2 = \xi(J_1 \times J_2)$. Then

$$\frac{1}{|I_1 \times I_2|} \|f\|_{L_1(I_1, L_\mathcal{B}(I_2))} = \frac{1}{|J_1 \times J_2|} \|f \circ \xi\|_{L_1(J_1, L_\mathcal{B}(J_2))}, \qquad \forall f \in L_1\left(I_1, L_\mathcal{B}(I_2)\right).$$

Proof.

$$\begin{aligned}\frac{1}{|I_1 \times I_2|} \|f\|_{L_1(I_1, L_\mathcal{B}(I_2))} &= \frac{1}{|I_1 \times I_2|} \int_{I_1} \|f(y_1, \cdot)\|^{(\mathrm{av})}_{\mathcal{B}, I_2}\, dy_1 \\ &= \frac{R_1}{|I_1 \times I_2|} \int_{J_1} \|f(R_1 x_1 + x_1^0, \cdot)\|^{(\mathrm{av})}_{\mathcal{B}, I_2}\, dx_1 \\ &= \frac{R_1 R_2}{|I_1 \times I_2|} \int_{J_1} \|f(R_1 x_1 + x_1^0, R_2 \cdot + x_2^0)\|^{(\mathrm{av})}_{\mathcal{B}, J_2}\, dx_1 \\ &= \frac{1}{|J_1 \times J_2|} \|f \circ \xi\|_{L_1(J_1, L_\mathcal{B}(J_2))}\end{aligned}$$

(see [29, Lemma 1]). ∎

Lemma 7.4. (Cf. [29, Lemma 3]) *Let rectangles $I_{1,k} \times I_{2,k}$, $k = 1, \ldots, n$ be pairwise disjoint subsets of $I_1 \times I_2$. Then*

$$\sum_{k=1}^{n} \|f\|_{L_1\left(I_{1,k}, L_\mathcal{B}(I_{2,k})\right)} \le \|f\|_{L_1(I_1, L_\mathcal{B}(I_2))}, \quad \forall f \in L_1\left(I_1, L_\mathcal{B}(I_2)\right). \tag{58}$$

Proof. Let us label the endpoints of the intervals I_1 and $I_{1,k}$, $k = 1, \ldots, n$ in the increasing order $a_1 < a_2 < \cdots < a_N$. Here $N \le 2(n+1)$ and N may be strictly less than $2(n+1)$ as some of these endpoints may coincide. Then

$$\begin{aligned}&\sum_{k=1}^{n} \|f\|_{L_1\left(I_{1,k}, L_\mathcal{B}(I_{2,k})\right)} = \sum_{k=1}^{n} \int_{I_{1,k}} \|f(x_1, \cdot)\|^{(\mathrm{av})}_{\mathcal{B}, I_{2,k}}\, dx_1 \\ &= \sum_{k=1}^{n} \sum_{(a_j, a_{j+1}) \subseteq I_{1,k}} \int_{a_j}^{a_{j+1}} \|f(x_1, \cdot)\|^{(\mathrm{av})}_{\mathcal{B}, I_{2,k}}\, dx_1 \\ &= \sum_{j=1}^{N-1} \int_{a_j}^{a_{j+1}} \sum_{\{k:\ I_{1,k} \supseteq (a_j, a_{j+1})\}} \|f(x_1, \cdot)\|^{(\mathrm{av})}_{\mathcal{B}, I_{2,k}}\, dx_1 \\ &\le \sum_{j=1}^{N-1} \int_{a_j}^{a_{j+1}} \|f(x_1, \cdot)\|^{(\mathrm{av})}_{\mathcal{B}, I_2}\, dx_1 = \int_{I_1} \|f(x_1, \cdot)\|^{(\mathrm{av})}_{\mathcal{B}, I_2}\, dx_1 = \|f\|_{L_1(I_1, L_\mathcal{B}(I_2))}.\end{aligned}$$

The inequality above follows from Lemma 3 in [29] and from the implication

$$I_{1,k} \supseteq (a_j, a_{j+1}),\ I_{1,m} \supseteq (a_j, a_{j+1}),\ k \neq m \quad \Longrightarrow \quad I_{2,k} \cap I_{2,m} = \emptyset,$$

which holds because the rectangles $I_{1,k} \times I_{2,k}$, $k = 1, \dots, n$ are pairwise disjoint. $\square$

Let $Q := (0,1)^2$ and $\mathbb{I} := (0,1)$. We will also use the following notation:

$$w_S := \frac{1}{|S|} \int_S w(x)\, dx,$$

where $S \subseteq \mathbb{R}^2$ is a set of a finite positive two dimensional Lebesgue measure $|S|$.

Lemma 7.5. (Cf. [29, Lemma 2]) *There exists $C_{14} > 0$ such that for any nonempty open intervals $I_1, I_2 \subseteq \mathbb{R}$ of lengths R_1 and R_2 respectively, any $w \in W_2^1(I_1 \times I_2) \cap C\left(\overline{I_1 \times I_2}\right)$ with $w_{I_1 \times I_2} = 0$, and any $V \in L_1\left(I_1, L_{\mathcal{B}}(I_2)\right)$, $V \geq 0$ the following inequality holds:*

$$\begin{aligned} &\int_{I_1 \times I_2} V(x) |w(x)|^2\, dx \\ &\leq C_{14} \max\left\{\frac{R_1}{R_2}, \frac{R_2}{R_1}\right\} \|V\|_{L_1(I_1, L_{\mathcal{B}}(I_2))} \int_{I_1 \times I_2} |\nabla w(x)|^2\, dx. \end{aligned} \tag{59}$$

Proof. Let us start with the case $I_1 = I_2 = \mathbb{I}$, $R_1 = R_2 = 1$. There exists $C_{15} > 0$ such that

$$\sup_{x_1 \in \mathbb{I}} \left\|w^2(x_1, \cdot)\right\|_{\mathcal{A}, \mathbb{I}} \leq C_{15} \|w\|^2_{W_2^1(Q)}, \quad \forall w \in W_2^1(Q) \cap C\left(\overline{Q}\right)$$

(see (8)). This can be proved by applying the trace theorem (see, e.g., [1, Theorems 4.32 and 7.53]) and then using the Yudovich–Pohozhaev–Trudinger embedding theorem for $H_2^{1/2}$ (see [12], [27] and [24, Lemma 1.2.4], [1, 8.25]) or, in one go, by applying a trace inequality of the Yudovich–Pohozhaev–Trudinger type (see [23, Corollary 11.8/2]; a sharp result can be found in [8]).

Next, we use the Poincaré inequality (see, e.g., [23, 1.11.1]): there exists $C_{16} > 0$ such that

$$\int_Q (w(x) - w_Q)^2\, dx = \inf_{a \in \mathbb{R}} \int_Q (w(x) - a)^2\, dx \leq C_{16} \int_Q |\nabla w(x)|^2\, dx, \qquad \forall w \in W_2^1(Q).$$

Hence

$$\sup_{x_1\in\mathbb{I}} \left\| w^2(x_1,\cdot)\right\|_{\mathcal{A},\mathbb{I}} \le C_{14} \int_Q |\nabla w(x)|^2\, dx, \quad \forall w \in W_2^1(Q) \cap C\left(\overline{Q}\right),\ w_Q = 0,$$

where $C_{14} = C_{15}(C_{16}+1)$. Now, the Hölder inequality (see [18, Theorem 9.3]) implies

$$\begin{aligned}\int_Q V(x)|w(x)|^2\, dx &\le \|V\|_{L_1(\mathbb{I},L_{\mathcal{B}}(\mathbb{I}))} \sup_{x_1\in\mathbb{I}} \left\| w^2(x_1,\cdot)\right\|_{\mathcal{A},\mathbb{I}} \\ &\le C_{14}\|V\|_{L_1(\mathbb{I},L_{\mathcal{B}}(\mathbb{I}))} \int_Q |\nabla w(x)|^2\, dx,\end{aligned}$$

i.e. (59) holds for $I_1 = I_2 = \mathbb{I}$.
Consider now arbitrary intervals $I_1, I_2 \subseteq \mathbb{R}$. Then $I_1 \times I_2 = \xi(Q)$, where ξ is the affine mapping from Lemma 7.3, and (59) can be derived from the above inequality with the help of Lemma 7.3 and the inequality

$$\int_Q |\nabla(w\circ\xi)(x)|^2\, dx \le \max\left\{\frac{R_1}{R_2}, \frac{R_2}{R_1}\right\} \int_{I_1\times I_2} |\nabla w(y)|^2\, dy$$

(see the the proof of Lemma 2 in [29]). □

Lemma 7.6. (Cf. [29, Theorem 1]) *For any $V \in L_1(\mathbb{I}, L_{\mathcal{B}}(\mathbb{I}))$, $V \ge 0$ and any $n \in \mathbb{N}$ there exists a finite cover of Q by rectangles $S_k = I_{1,k} \times I_{2,k}$, $k = 1, \dots, n_0$ such that $n_0 \le n$ and*

$$\int_Q V(x)|w(x)|^2\, dx \le C_{17} n^{-1} \|V\|_{L_1(\mathbb{I},L_{\mathcal{B}}(\mathbb{I}))} \int_Q |\nabla w(x)|^2\, dx \tag{60}$$

for all $w \in W_2^1(Q) \cap C\left(\overline{Q}\right)$ with $w_{S_k} = 0$, $k = 1, \dots, n_0$, where the constant C_{17} does not depend on V.

Proof. According to the Besicovitch covering lemma (see, e.g., [13, Ch. I. Theorem 1.1]), there exists a constant $\nu \in \mathbb{N}$ such that any cover $\{\Delta_x\}_{x\in\overline{Q}}$ of $\overline{Q}$ by closed squares Δ_x centered at x has a countable or a finite subcover that can be split into ν families in such a way that any two squares belonging to the same family are disjoint.
We can assume that $n > \nu$, as otherwise one could take $n_0 = 1$, $S_k = Q$ and get (60) with $C_{17} \ge \nu C_{14}$ directly from (59). For any $x \in \overline{Q}$ there exists a closed square $\Delta_x = I^0_{1,x} \times I^0_{2,x}$ centered at x such that

$$\|V\|_{L_1(I_{1,x},L_{\mathcal{B}}(I_{2,x}))} = \nu n^{-1} \|V\|_{L_1(\mathbb{I},L_{\mathcal{B}}(\mathbb{I}))},$$

where

$$I_{i,x} := I^0_{i,x} \cap [0,1],\ i = 1, 2, \quad \text{i.e.} \quad I_{1,x} \times I_{2,x} = \Delta_x \cap \overline{Q}$$

(see [29, Lemma 4]). Let $\{\Delta_{x_k}\}$ be the subcover from the Besicovitch covering lemma and let $S_k = I_{1,k} \times I_{2,k} := I_{1,x_k} \times I_{2,x_k} = \Delta_{x_k} \cap \overline{Q}$. Then $\Xi := \{S_k\}$ is also a cover of $\overline{Q}$ and, like $\{\Delta_{x_k}\}$, it can be split into ν families Ξ_l, $l = 1, \dots, \nu$, consisting of pairwise disjoint elements. Lemma 7.4 implies

$$\nu n^{-1} \|V\|_{L_1(\mathbb{I}, L_{\mathcal{B}}(\mathbb{I}))} \operatorname{card} \Xi_l = \sum_{S_k \in \Xi_l} \|V\|_{L_1\left(I_{1,k}, L_{\mathcal{B}}(I_{2,k})\right)} \le \|V\|_{L_1(\mathbb{I}, L_{\mathcal{B}}(\mathbb{I}))}.$$

Hence $\operatorname{card} \Xi_l \le n/\nu$ and

$$n_0 := \operatorname{card} \Xi = \sum_{l=1}^{\nu} \operatorname{card} \Xi_l \le \nu n/\nu = n.$$

Take any $w \in W^1_2(Q) \cap C\left(\overline{Q}\right)$ with $w_{S_k} = 0$, $k = 1, \dots, n_0$. Since the centre of the square Δ_{x_k} belongs to $\overline{Q}$, the ratio of the side lengths of the rectangle $S_k = \Delta_{x_k} \cap \overline{Q}$ is between $1/2$ and 2. Hence it follows from Lemma 7.5 that

$$\begin{aligned}
&\int_Q V(x)|w(x)|^2\, dx \le \sum_{k=1}^{n_0} \int_{S_k} V(x)|w(x)|^2\, dx \\
&\le 2C_{14} \sum_{k=1}^{n_0} \|V\|_{L_1\left(I_{1,k}, L_{\mathcal{B}}(I_{2,k})\right)} \int_{S_k} |\nabla w(x)|^2\, dx \\
&= 2C_{14}\nu n^{-1} \|V\|_{L_1(\mathbb{I}, L_{\mathcal{B}}(\mathbb{I}))} \sum_{k=1}^{n_0} \int_{S_k} |\nabla w(x)|^2\, dx \\
&= 2C_{14}\nu n^{-1} \|V\|_{L_1(\mathbb{I}, L_{\mathcal{B}}(\mathbb{I}))} \sum_{l=1}^{\nu} \sum_{S_k \in \Xi_l} \int_{S_k} |\nabla w(x)|^2\, dx \\
&= 2C_{14}\nu n^{-1} \|V\|_{L_1(\mathbb{I}, L_{\mathcal{B}}(\mathbb{I}))} \sum_{l=1}^{\nu} \int_Q |\nabla w(x)|^2\, dx \\
&= C_{17} n^{-1} \|V\|_{L_1(\mathbb{I}, L_{\mathcal{B}}(\mathbb{I}))} \int_Q |\nabla w(x)|^2\, dx,
\end{aligned}$$

where $C_{17} := 2C_{14}\nu^2$. $\square$

Let

$$\begin{gathered}
\mathcal{E}_{V,0}[w] := \int_Q |\nabla w(x)|^2 dx - \int_Q V(x)|w(x)|^2 dx, \\
\operatorname{Dom}(\mathcal{E}_{V,0}) = W^1_2(Q) \cap L_2(Q, V(x)dx).
\end{gathered}$$

Lemma 7.7. (Cf. [29, Theorem 4])

$$N_-(\mathcal{E}_{V,0}) \le C_{17}\|V\|_{L_1(\mathbb{I},L_{\mathcal{B}}(\mathbb{I}))} + 1, \quad \forall V \ge 0,$$

where C_{17} is the same as in Lemma 7.6.

Proof. Let $n = \left[C_{17}\|V\|_{L_1(\mathbb{I},L_{\mathcal{B}}(\mathbb{I}))}\right] + 1$ in Lemma 7.6. Take any linear subspace $\mathcal{L} \subset \mathrm{Dom}\,(\mathcal{E}_{V,0})$ such that

$$\dim \mathcal{L} > \left[C_{17}\|V\|_{L_1(\mathbb{I},L_{\mathcal{B}}(\mathbb{I}))}\right] + 1.$$

Since $n_0 \le n$, there exists $w \in \mathcal{L} \setminus \{0\}$ such that $w_{S_k} = 0$, $k = 1, \dots, n_0$. Then

$$\begin{aligned}
\mathcal{E}_{V,0}[w] &= \int_Q |\nabla w(x)|^2 dx - \int_Q V(x)|w(x)|^2 dx \\
&\ge \int_Q |\nabla w(x)|^2 dx - \frac{C_{17}\|V\|_{L_1(\mathbb{I},L_{\mathcal{B}}(\mathbb{I}))}}{\left[C_{17}\|V\|_{L_1(\mathbb{I},L_{\mathcal{B}}(\mathbb{I}))}\right] + 1} \int_Q |\nabla w(x)|^2 dx \\
&\ge \int_Q |\nabla w(x)|^2 dx - \int_Q |\nabla w(x)|^2 dx = 0.
\end{aligned}$$

Hence

$$N_-(\mathcal{E}_V) \le \left[C_{17}\|V\|_{L_1(\mathbb{I},L_{\mathcal{B}}(\mathbb{I}))}\right] + 1 \le C_{17}\|V\|_{L_1(\mathbb{I},L_{\mathcal{B}}(\mathbb{I}))} + 1.$$

$\square$

Let $R > 0$, $I(R) := (R, eR)$, $\Omega(R) := \{x \in \mathbb{R}^2 : \ R < |x| < eR\}$ and

$$\begin{gathered}
\mathcal{E}_V^R[w] := \int_{\Omega(R)} |\nabla w(x)|^2 dx - \int_{\Omega(R)} V(x)|w(x)|^2 dx, \\
\mathrm{Dom}\,(\mathcal{E}_V^R) = \left\{w \in W_2^1\left(\Omega(R)\right) \cap L_2\left(\Omega(R), V(x)dx\right) : \ w_{\Omega(R)} = 0\right\}.
\end{gathered}$$

Lemma 7.8. (Cf. [29, Theorem 4′]) *There exists $C_{18} > 0$ such that*

$$N_-(\mathcal{E}_V^R) \le C_{18}\|V\|_{\mathcal{L}_1(I(R),L_{\mathcal{B}}(\mathbb{S}))}, \quad \forall V \ge 0, \quad \forall R > 0. \tag{61}$$

Proof. We start with the case $R = 1$. Let $\mathbb{S}_+ := (0, \pi)$, $\mathbb{S}_- := (-\pi, 0)$,

$$\begin{gathered}
\Omega_\pm := \{(r\cos\vartheta, r\sin\vartheta) \in \mathbb{R}^2 : \ 1 < r < e, \ \vartheta \in \mathbb{S}_\pm\} \\
\mathcal{E}_V^\pm[w] := \int_{\Omega_\pm} |\nabla w(x)|^2 dx - \int_{\Omega_\pm} V(x)|w(x)|^2 dx, \\
\mathrm{Dom}\,(\mathcal{E}_V^\pm) = W_2^1\left(\Omega_\pm\right) \cap L_2\left(\Omega_\pm, V(x)dx\right).
\end{gathered}$$

Then by the variational principle,

$$N_-(\mathcal{E}_V^1) \le N_-(\mathcal{E}_V^+) + N_-(\mathcal{E}_V^-). \tag{62}$$

(see, e.g., [10, Lemma 3.5]). Consider the diffeomorphisms $\varphi_\pm : Q \to \Omega_\pm$,

$$\varphi_\pm(y_1, y_2) = \left(\big(1 + (e-1)y_1\big)\cos(\pm\pi y_2), \big(1 + (e-1)y_1\big)\sin(\pm\pi y_2)\right).$$

The polar coordinates of $\varphi_\pm(y_1, y_2)$ are $r = 1 + (e-1)y_1$, $\vartheta = \pm\pi y_2$.
Let

$$\hat{V}_\pm(y) := V(\varphi_\pm(y)), \quad \tilde{w}(y) := w(\varphi_\pm(y)),\ w \in \mathrm{Dom}\,(\mathcal{E}_V^\pm).$$

There exist absolute constants $C_{19} > 0$ and $C_{20} > 0$ such that

$$\int_{\Omega_\pm} |\nabla w(x)|^2 dx \ge \frac{1}{C_{19}} \int_Q |\nabla \tilde{w}(y)|^2 dy,$$
$$\int_{\Omega_\pm} V(x)|w(x)|^2 dx \le C_{20} \int_Q \hat{V}_\pm(y)|\tilde{w}(y)|^2 dy.$$

Then Lemma 7.7 implies

$$N_-(\mathcal{E}_V^\pm) \le N_-(\mathcal{E}_{C_{19}C_{20}\hat{V}_\pm, 0}) \le C_{17}C_{19}C_{20} \left\|\hat{V}_\pm\right\|_{L_1(\mathbb{I}, L_\mathcal{B}(\mathbb{I}))} + 1.$$

Using [29, Lemma 1] one gets

$$\left\|\hat{V}_\pm\right\|_{L_1(\mathbb{I}, L_\mathcal{B}(\mathbb{I}))} = \int_\mathbb{I} \left\|\hat{V}_\pm(y_1, \cdot)\right\|^{(\mathrm{av})}_{\mathcal{B},\mathbb{I}} dy_1 = \frac{1}{\pi(e-1)} \int_1^e \|V(r, \cdot)\|^{(\mathrm{av})}_{\mathcal{B},\mathbb{S}_\pm} dr.$$

Now it follows from (62), Lemma 2.1 and [29, Lemma 3] that

$$\begin{aligned} N_-(\mathcal{E}_V^1) &\le C_{21} \int_1^e \left(\|V(r,\cdot)\|^{(\mathrm{av})}_{\mathcal{B},\mathbb{S}_+} + \|V(r,\cdot)\|^{(\mathrm{av})}_{\mathcal{B},\mathbb{S}_-}\right) dr + 2 \\ &\le C_{21} \int_1^e \|V(r,\cdot)\|^{(\mathrm{av})}_{\mathcal{B},\mathbb{S}} dr + 2 \le C_{21} \int_1^e \|V(r,\cdot)\|^{(\mathrm{av})}_{\mathcal{B},\mathbb{S}} r dr + 2 \\ &\le 2\pi C_{21} \|V\|_{\mathcal{L}_1(I(1), L_\mathcal{B}(\mathbb{S}))} + 2, \end{aligned} \tag{63}$$

where $C_{21} = \frac{C_{17}C_{19}C_{20}}{\pi(e-1)}$.
Using the Yudovich–Pohozhaev–Trudinger embedding theorem, the Poincaré inequality and the Hölder inequality for Orlicz spaces as in the proof of Lemma 7.5 one can prove the existence of a constant $C_{22} > 0$ such that

$$\int_{\Omega(1)} V(x)|w(x)|^2 dx \le C_{22} \|V\|_{\mathcal{L}_1(I(1), L_\mathcal{B}(\mathbb{S}))} \int_{\Omega(1)} |\nabla w(x)|^2 dx, \quad \forall w \in \mathrm{Dom}\,(\mathcal{E}_V^1).$$

If $\|V\|_{\mathcal{L}_1(I(1), L_\mathcal{B}(\mathbb{S}))} \le 1/C_{22}$, then

$$\mathcal{E}_V^1[w] \ge 0, \quad \forall w \in \mathrm{Dom}\,(\mathcal{E}_V^1),$$

i.e. $N_-(\mathcal{E}_V^1) = 0$. Combining this with (63) one gets the existence of a constant $C_{18} > 0$ for which (61) holds in the case $R = 1$. The case of a general R is reduced to $R = 1$ in the standard way with the help of the scaling $x \mapsto Rx$ (cf. the proof of Theorem 4′ in [29]). $\square$

Proof of Theorem 7.1. The theorem is proved in the same way as Theorem 6.1 (see (26), (40) and (42)). One only needs to use Lemma 7.8 instead of Theorem 4′ of [29]. $\square$

Remark 7.9. It has been observed by A. Laptev that interchanging the rôles of the variables y_1 and y_2 in the proof of Lemma 7.8 one can get an analogue of (61) where the $L_{\mathcal{B}}$ norm is taken with respect to the radial variable, while the L_1 norm is taken with respect to the angular one. Then using the Birman-Solomyak method exactly as in the proofs of Theorems 6.1 and 7.1, one shows the existence of constants $C_{23} > 0$ and $c > 0$ such that

$$N_-(\mathcal{E}_V) \le 1 + 4 \sum_{A_n > 1/4} \sqrt{A_n} + C_{23} \sum_{\mathcal{G}_n > c} \mathcal{G}_n\,, \quad \forall V \ge 0, \tag{64}$$

where

$$\mathcal{G}_n := e^n \int_{\mathbb{S}} \|V(\cdot, \vartheta)\|^{(\mathrm{av})}_{\mathcal{B},\mathcal{I}_n}\, d\vartheta\,, \quad n \in \mathbb{Z} \tag{65}$$

(cf. (50), (51)), and the factor e^n in front of the integral ensures the correct scaling under the mapping $x \mapsto e^n x$.

8 Comparison of various estimates

Let us start with comparing (11) and (46). Note first of all that (46) is equivalent to

$$N_-(\mathcal{E}_V) \le 1 + C_{24} \left(\|(\mathbf{A}_n)_{n\ge 1}\|_{1,\infty} + \sum_{n=0}^{\infty} \mathbf{B}_n \right), \quad \forall V \ge 0, \tag{66}$$

where we have dropped the term $\mathbf{A}_0$. Indeed, the Hölder inequality (see [28, §3.3, (17)]) implies

$$\mathbf{A}_0 = \int_{|x|\le e} V(x)|\ln|x||\,dx \le \|V\|_{\mathcal{B},\mathbf{\Omega}_0}\||\ln|\cdot|\||_{(\mathcal{A},\mathbf{\Omega}_0)},$$

$$\int_{\mathbf{\Omega}_0} \mathcal{A}\left(|\ln|x||\right)\,dx \le \int_{|x|\le 1} e^{\ln\frac{1}{|x|}}\,dx + \int_{1<|x|\le e} e^{\ln|x|}\,dx$$
$$= \int_{|x|\le 1} \frac{1}{|x|}\,dx + \int_{1<|x|\le e} |x|\,dx$$
$$= \int_{-\pi}^{\pi}\int_0^1 1\,drd\vartheta + \int_{-\pi}^{\pi}\int_1^e r^2\,drd\vartheta = \frac{2\pi}{3}\left(e^3+2\right) =: C_{25}.$$

Hence

$$\||\ln|\cdot|\||_{(\mathcal{A},\mathbf{\Omega}_0)} \le C_{25}$$

(see (4)) and

$$\mathbf{A}_0 \le C_{25}\|V\|_{\mathcal{B},\mathbf{\Omega}_0} \le C_{25}\|V\|^{(\mathrm{av})}_{\mathcal{B},\mathbf{\Omega}_0} = C_{25}\mathbf{B}_0. \tag{67}$$

It is clear that

$$\|(\mathbf{A}_n)_{n\ge1}\|_{1,\infty} \le \|(\mathbf{A}_n)_{n\ge1}\|_1 = \sum_{n=1}^{\infty}\int_{U_n} V(x)\ln|x|\,dx$$
$$= \int_{|x|>e} V(x)\ln|x|\,dx \le \int_{\mathbb{R}^2} V(x)\ln(1+|x|)\,dx. \tag{68}$$

It is also easy to see that if

$$V(x) = \begin{cases} \frac{1}{2\pi|x|^2(\ln|x|)^2\ln\ln|x|}, & |x|\ge e^2, \\ 0, & x| < e^2, \end{cases} \tag{69}$$

then $\mathbf{A}_n = \ln\left(1+\frac{1}{n-1}\right) = \frac{1}{n} + O\left(n^{-2}\right)$, $n \ge 2$, and the left-hand side of (68) is finite, while the right-hand side is infinite.
On the other hand,

$$\sum_{n=0}^{\infty}\mathbf{B}_n \ge \|V\|_{\mathcal{B},\mathbf{\Omega}_0} + \sum_{n=1}^{\infty}\|V\|_{\mathcal{B},\Omega_n} \ge \|V\|_{\mathcal{B},\mathbb{R}^2}, \tag{70}$$

where the last estimate follows from the triangle inequality applied to $f = f\chi_{\mathbf{\Omega}_0} + \sum_{n=1}^{\infty} f\chi_{\Omega_n}$ with $\chi_\Omega(x) := 1$ for $x\in\Omega$ and $\chi_\Omega(x) := 0$ for $x \notin \Omega$. It is not difficult to show that there exist a V for which the left-hand side of

(70) is infinite while the middle term is finite, and a V for which the middle term is infinite while the right-hand side is finite. Indeed, let $V(x) = \frac{1}{|x|^2}$ for $|x| \ge e$ and $V(x) = 0$ for $|x| < e$. Then using [29, (7)] and the notation from Remark 6.3 one gets

$$\sum_{n=1}^{\infty} \mathbf{B}_n = \sum_{n=1}^{\infty} \mathcal{B}_n \ge \sum_{n=1}^{\infty} e^{-2(n+1)} \|1\|^{(\mathrm{av})}_{\mathcal{B},\Omega_n} = \sum_{n=1}^{\infty} e^{-2(n+1)} |\Omega_n| \|1\|_{\mathcal{B},\Delta}$$
$$= \|1\|_{\mathcal{B},\Delta} \sum_{n=1}^{\infty} \pi \left(1 - \frac{1}{e^2}\right) = +\infty.$$

On the other hand, [18, (9.11)] and Lemma 2.3 imply

$$\sum_{n=1}^{\infty} \|V\|_{\mathcal{B},\Omega_n} \le \sum_{n=1}^{\infty} e^{-2n} \|1\|_{\mathcal{B},\Omega_n} = \sum_{n=1}^{\infty} e^{-2n} |\Omega_n| \mathcal{A}^{-1}\left(\frac{1}{|\Omega_n|}\right)$$
$$\le \sum_{n=1}^{\infty} e^{-2n} \sqrt{2|\Omega_n|} = \sum_{n=1}^{\infty} e^{-n} \sqrt{2\pi(e^2 - 1)} < +\infty.$$

Take now $V(x) = \frac{1}{|x| \ln |x|}$ for $|x| \ge e$ and $V(x) = 0$ for $|x| < e$. Then

$$\sum_{n=1}^{\infty} \|V\|_{\mathcal{B},\Omega_n} \ge \sum_{n=1}^{\infty} \frac{e^{-(n+1)}}{n+1} \|1\|_{\mathcal{B},\Omega_n} = \sum_{n=1}^{\infty} \frac{e^{-(n+1)}}{n+1} |\Omega_n| \mathcal{A}^{-1}\left(\frac{1}{|\Omega_n|}\right)$$
$$\ge \sum_{n=1}^{\infty} \frac{e^{-(n+1)}}{n+1} \sqrt{\frac{2}{e} |\Omega_n|} = \sum_{n=1}^{\infty} \frac{1}{n+1} \sqrt{\frac{2\pi}{e}\left(1 - \frac{1}{e^2}\right)} = +\infty,$$

while

$$\int_{\mathbb{R}^2} \mathcal{B}(V(x))\, dx = \int_{|x| \ge e} \mathcal{B}\left(\frac{1}{|x| \ln |x|}\right) dx = 2\pi \int_{r \ge e} \mathcal{B}\left(\frac{1}{r \ln r}\right) r dr$$
$$\le \pi \int_{r \ge e} \frac{1}{r \ln^2 r}\, dr = \pi$$

(see Lemma 2.3) and hence $\|V\|_{\mathcal{B},\mathbb{R}^2} < \infty$.

The above discussion of (68) and (70) may give one an impression that neither of (11) and (46) is stronger than the other one. It turns out that this is not the case, and it follows from (67), (68) and Lemmas 2.1, 2.5 and 8.1 (see

below) that (46) is actually stronger than (11). Indeed,

$$
\begin{aligned}
&\|(\mathbf{A}_n)_{n\ge 0}\|_{1,\infty} + \sum_{n=0}^{\infty} \mathbf{B}_n \le \mathrm{const}\left(\|(\mathbf{A}_n)_{n\ge 1}\|_{1,\infty} + \sum_{n=0}^{\infty} \mathbf{B}_n\right)\\
&\le \mathrm{const}\Big(\int_{\mathbb{R}^2} V(x)\ln(1+|x|)\,dx + \pi e^2 \|V\|_{\mathcal{B},\mathbf{\Omega}_0}\\
&+ \sum_{n=1}^{\infty} \|V\|_{\mathcal{B},\Omega_n} + \sum_{n=1}^{\infty} \ln\left(\frac{7}{2}\,|\Omega_n|\right)\int_{\Omega_n} V(x)\,dx\Big) \qquad (71)\\
&\le \mathrm{const}\left(\int_{\mathbb{R}^2} V(x)\ln(1+|x|)\,dx + \|V\|_{\mathcal{B},\mathbb{R}^2} + \int_{|x|>e} V(x)\ln\ln|x|\,dx\right)\\
&\le \mathrm{const}\left(\|V\|_{\mathcal{B},\mathbb{R}^2} + \int_{\mathbb{R}^2} V(x)\ln(1+|x|)\,dx\right), \quad \forall V\ge 0.
\end{aligned}
$$

Note that for V given by (69) one has

$$
\begin{aligned}
\sum_{n=0}^{\infty} \mathbf{B}_n = \sum_{n=2}^{\infty} \mathcal{B}_n \le \sum_{n=2}^{\infty} \frac{1}{2\pi e^{2n} n^2 \ln n}\|1\|^{(\mathrm{av})}_{\mathcal{B},\Omega_n} &= \|1\|_{\mathcal{B},\Delta} \sum_{n=2}^{\infty} \frac{|\Omega_n|}{2\pi e^{2n} n^2 \ln n}\\
&= \|1\|_{\mathcal{B},\Delta} \sum_{n=2}^{\infty} \frac{e^2-1}{2n^2\ln n} < +\infty.
\end{aligned}
$$

Hence the right-hand side of (46) is finite for this V while the right-hand side of (11) is infinite. So, (46) is strictly stronger than (11).

Lemma 8.1. *There exists $C_{26} > 0$ such that*

$$
\sum_{n=1}^{\infty} \|V\|_{\mathcal{B},\Omega_n} \le C_{26}\left(\|V\|_{\mathcal{B},\mathbb{R}^2\setminus\mathbf{\Omega}_0} + \int_{|x|>e} V(x)\ln\ln|x|\,dx\right), \quad \forall V \ge 0
$$

(see (41), (47)*).*

Proof. Suppose first that $\|V\|_{(\mathcal{B},\mathbb{R}^2\setminus\mathbf{\Omega}_0)} = 1$ and let

$$
\alpha_n := \int_{\Omega_n} \mathcal{B}(V(x))\,dx, \quad \kappa_n := \|V\|_{(\mathcal{B},\Omega_n)}, \quad n\in\mathbb{N}.
$$

Then

$$
\sum_{n=1}^{\infty} \alpha_n = \sum_{n=1}^{\infty} \int_{\Omega_n} \mathcal{B}(V(x))\,dx = \int_{\mathbb{R}^2\setminus\mathbf{\Omega}_0} \mathcal{B}(V(x))\,dx = 1,
$$

$$
\alpha_n \le 1 \implies \kappa_n \le 1,
$$

and

$$
\begin{aligned}
1 = \int_{\Omega_n} \mathcal{B}\left(\frac{V(x)}{\kappa_n}\right) dx \le \int_{\Omega_n} \left(\frac{V(x)}{\kappa_n} + 2\frac{V(x)}{\kappa_n} \ln_+ \frac{V(x)}{\kappa_n}\right) dx \\
\le \frac{1}{\kappa_n} \int_{\Omega_n} (V(x) + 2V(x) \ln_+ V(x))\, dx + \frac{2}{\kappa_n} \ln \frac{1}{\kappa_n} \|V\|_{L_1(\Omega_n)} \\
\le \frac{4}{\kappa_n} \alpha_n + \frac{1}{\kappa_n} \left(1 + 2 \ln \frac{1}{\kappa_n}\right) \|V\|_{L_1(\Omega_n)}
\end{aligned}
$$

(see Lemma 2.2). Hence

$$
\kappa_n \le 4\alpha_n + \left(1 + 2 \ln \frac{1}{\kappa_n}\right) \|V\|_{L_1(\Omega_n)}
$$

and

$$
\begin{aligned}
\sum_{n=1}^{\infty} \|V\|_{\mathcal{B},\Omega_n} \le 2 \sum_{n=1}^{\infty} \kappa_n = 2 \sum_{\kappa_n \le 1/n^2} \kappa_n + 2 \sum_{\kappa_n > 1/n^2} \kappa_n \\
\le 2 \sum_{n=1}^{\infty} \frac{1}{n^2} + 8 \sum_{n=1}^{\infty} \alpha_n + 2 \sum_{n=1}^{\infty} (1 + 4 \ln n) \|V\|_{L_1(\Omega_n)} \\
= \frac{\pi^2}{3} + 8 + 2 \sum_{n=1}^{\infty} (1 + 4 \ln n) \int_{e^n < |x| < e^{n+1}} V(x)\, dx \\
\le \frac{\pi^2}{3} + 8 + 2 \int_{|x| > e} V(x)(1 + 4 \ln \ln |x|)\, dx \\
\le C_{26} \left(\|V\|_{\mathcal{B},\mathbb{R}^2 \setminus \mathbf{\Omega}_0} + \int_{|x| > e} V(x) \ln \ln |x|\, dx\right)
\end{aligned}
$$

(see (3)). The case of a general V is reduced to $\|V\|_{(\mathcal{B},\mathbb{R}^2 \setminus \mathbf{\Omega}_0)} = 1$ by the scaling $V \mapsto tV$, $t > 0$. $\square$

Let us now show that each of (52) and (64) implies (45) (with a different c).

Suppose $\|V\|_{(\mathcal{B},\Omega_0)} = 1$. It follows from (3), (6) and Lemma 2.1 that

$$\mathcal{D}_0 = \int_1^e \|V(r,\cdot)\|_{\mathcal{B},\mathbb{S}}\, rdr \le 2\int_1^e \left(1 + \int_{\mathbb{S}} \mathcal{B}(V(r,\vartheta))\, d\vartheta\right) rdr$$
$$\le (e^2-1) + 2\int_{\Omega_0} \mathcal{B}(V(x))\, dx = e^2 + 1 = (e^2+1)\|V\|_{(\mathcal{B},\Omega_0)}$$
$$\le (e^2+1)\|V\|_{\mathcal{B},\Omega_0} \le (e^2+1)\|V\|^{(\mathrm{av})}_{\mathcal{B},\Omega_0} = (e^2+1)\mathcal{B}_0,$$

$$\mathcal{G}_0 = \int_{\mathbb{S}} \|V(\cdot,\vartheta)\|^{(\mathrm{av})}_{\mathcal{B},\mathcal{I}_0}\, d\vartheta \le 2(e-1)\int_{\mathbb{S}} \left(1 + \int_1^e \mathcal{B}(V(r,\vartheta))\, dr\right) d\vartheta$$
$$\le 2(e-1)\left(2\pi + \int_{\Omega_0} \mathcal{B}(V(x))\, dx\right) = 2(e-1)(2\pi+1)$$
$$= 2(e-1)(2\pi+1)\|V\|_{(\mathcal{B},\Omega_0)} \le 2(e-1)(2\pi+1)\mathcal{B}_0 =: C_{27}\mathcal{B}_0.$$

The scaling $V \mapsto tV$, $t > 0$ allows one to extend the inequalities $\mathcal{D}_0 \le (e^2+1)\mathcal{B}_0$ and $\mathcal{G}_0 \le C_{27}\mathcal{B}_0$ to arbitrary $V \ge 0$. Using the scaling $x \mapsto e^n x$ of the independent variable and [29, Lemma 1], one gets $\mathcal{D}_n \le (e^2+1)\mathcal{B}_n$ and $\mathcal{G}_n \le C_{27}\mathcal{B}_n$, $\forall n \in \mathbb{Z}$. Hence

$$\sum_{\{n\in\mathbb{Z}:\ \mathcal{D}_n > c\}} \mathcal{D}_n \le (e^2+1) \sum_{\{n\in\mathbb{Z}:\ \mathcal{D}_n > c\}} \mathcal{B}_n \le (e^2+1) \sum_{\{n\in\mathbb{Z}:\ \mathcal{B}_n > c/(e^2+1)\}} \mathcal{B}_n\,,$$

$$\sum_{\{n\in\mathbb{Z}:\ \mathcal{G}_n > c\}} \mathcal{G}_n \le C_{27} \sum_{\{n\in\mathbb{Z}:\ \mathcal{G}_n > c\}} \mathcal{B}_n \le C_{27} \sum_{\{n\in\mathbb{Z}:\ \mathcal{B}_n > c/C_{27}\}} \mathcal{B}_n\,.$$

Similarly,

$$\|V\|_{\mathcal{L}_1(\mathbb{R}_+, L_{\mathcal{B}}(\mathbb{S}))} = \sum_{n\in\mathbb{Z}} \mathcal{D}_n \le (e^2+1)\sum_{n\in\mathbb{Z}} \mathcal{B}_n,$$

and (53) implies (46).
Putting together what we have obtained so far, we get the diagram below. For the convenience of the reader, we precede the diagram with a list of the estimates discussed above:

$$N_-(\mathcal{E}_V) \le C\left(\|V\|_{\mathcal{B},\mathbb{R}^2} + \int_{\mathbb{R}^2} V(x)\ln(1+|x|)\, dx\right) + 1, \tag{11}$$

$$N_-(\mathcal{E}_V) \le C_6\left(\int_{\mathbb{R}^2} V(x)\ln(2+|x|)\, dx + \int_{\mathbb{R}^2} V_*(|x|)\ln_+ \frac{1}{|x|}\, dx\right) + 1, \tag{21}$$

$$N_-(\mathcal{E}_V) \le c_1\int_{\mathbb{R}^2} V_*(|x|)\ln_+ \frac{1}{|x|}\, dx + c_2\int_{\mathbb{R}^2} V(x)\ln_+|x|\, dx + c_3\int_{\mathbb{R}^2} V(x)\, dx + 1, \tag{22}$$

$$N_-(\mathcal{E}_V) \le 1 + 4\sum_{A_n > 1/4} \sqrt{A_n} + C_7\sum_{\mathcal{B}_n > c} \mathcal{B}_n, \tag{45}$$

$$N_-(\mathcal{E}_V) \le 1 + C_8 \left(\|(\mathbf{A}_n)_{n\ge 0}\|_{1,\infty} + \sum_{n=0}^{\infty} \mathbf{B}_n \right), \tag{46}$$

$$N_-(\mathcal{E}_V) \le 1 + C_7 \sum_{\{n\in\mathbb{Z}:\ A_n > c\}} \sqrt{A_n} + C_7 \sum_{\{n\in\mathbb{Z}:\ B_n > c\}} B_n, \tag{49}$$

$$N_-(\mathcal{E}_V) \le 1 + 4 \sum_{A_n > 1/4} \sqrt{A_n} + C_9 \sum_{\mathcal{D}_n > c} \mathcal{D}_n, \tag{52}$$

$$N_-(\mathcal{E}_V) \le 1 + C_{10} \left(\|(A_n)_{n\in\mathbb{Z}}\|_{1,\infty} + \|V\|_{\mathcal{L}_1(\mathbb{R}_+, L_{\mathcal{B}}(\mathbb{S}))} \right), \tag{53}$$

$$N_-(\mathcal{E}_V) \le 1 + C_{11} \left(\|(A_n)_{n\in\mathbb{Z}}\|_{1,\infty} + \int_{\mathbb{R}_+} \left(\int_{\mathbb{S}} |V(r,\vartheta)|^p d\vartheta \right)^{1/p} r dr \right), \tag{54}$$

$$N_-(\mathcal{E}_V) \le 1 + C_{12} \left(\|(A_n)_{n\in\mathbb{Z}}\|_{1,\infty} + \int_{\mathbb{R}_+} \left(\int_{\mathbb{S}} |V_{\mathcal{N}}(r,\vartheta)|^p d\vartheta \right)^{1/p} r dr \right), \tag{55}$$

$$N_-(\mathcal{E}_V) \le 1 + C_{13} \left(\|(A_n)_{n\in\mathbb{Z}}\|_{1,\infty} + \|V_{\mathcal{N}}\|_{\mathcal{L}_1(\mathbb{R}_+, L_{\mathcal{B}}(\mathbb{S}))} \right), \tag{56}$$

$$N_-(\mathcal{E}_V) \le 1 + 4 \sum_{A_n > 1/4} \sqrt{A_n} + C_{23} \sum_{\mathcal{G}_n > c} \mathcal{G}_n\,, \tag{64}$$

where

$$A_0 := \int_{U_0} V(x)\, dx, \quad A_n := \int_{U_n} V(x) |\ln |x||\, dx, \ n \ne 0,$$

$$\mathcal{B}_n := \|V\|^{(\mathrm{av})}_{\mathcal{B},\Omega_n}, \quad B_n := \left(\int_{\Omega_n} V^p(x) |x|^{2(p-1)}\, dx \right)^{1/p},$$

$$\mathbf{A}_n = A_n,\ \mathbf{B}_n = \mathcal{B}_n, \quad n \in \mathbb{N},$$

$$\mathbf{A}_0 := \int_{\mathbf{\Omega}_0} V(x) |\ln |x||\, dx, \quad \mathbf{B}_0 = \|V\|^{(\mathrm{av})}_{\mathcal{B},\mathbf{\Omega}_0}, \quad \mathbf{\Omega}_0 = \{x \in \mathbb{R}^2 :\ |x| \le e\},$$

$$\mathcal{D}_n := \|V\|_{\mathcal{L}_1(\mathcal{I}_n, L_{\mathcal{B}}(\mathbb{S}))}, \quad \mathcal{G}_n := e^n \int_{\mathbb{S}} \|V(\cdot,\vartheta)\|^{(\mathrm{av})}_{\mathcal{B},\mathcal{I}_n}\, d\vartheta\,, \quad \mathcal{I}_n := (e^n, e^{n+1}), \quad n \in \mathbb{Z},$$

$1 < p < \infty$, and U_n, Ω_n are defined in (38), (41). Estimate (22) was conjectured by N.N. Khuri, A. Martin and T.T. Wu in [17], (46) was proved by M. Solomyak in [29], (49) was obtained by A. Grigor'yan and N. Nadirashvili in [10], (55) was proved by A. Laptev and M. Solomyak in [21], and (64) was proposed by A. Laptev (see Remark 7.9).

$$\begin{array}{ccccccc}
 & & (52) & \Longrightarrow & \boxed{(53) \Longleftrightarrow (56)} & \Longrightarrow & \boxed{(54) \Longleftrightarrow (55)} \\
 & & \Downarrow & & \Downarrow & & \\
(64) & \Longrightarrow & (45) & \Longrightarrow & (46) & \Longrightarrow & \boxed{(11) \Longleftrightarrow (21) \Longleftrightarrow (22)} \\
 & & \Downarrow & & & & \\
 & & (49) & & & &
\end{array}$$

Our next task is to show that no other implication holds between the estimates in the above diagram. Suppose $V(r,\vartheta) = V_1(r)V_2(\vartheta)$. If $V_2 \equiv 1$, $V_1(r) = \frac{\alpha}{r^2(1+\ln^2 r)}$ and $\alpha > 0$ is sufficiently small, then the right-hand side of (49) equals 1 while $\|(A_n)_{n\in\mathbb{N}}\|_{1,\infty} = +\infty$ (see [10]). Hence (53) does not imply (49).
Suppose now $\operatorname{supp} V_1 \subseteq [1,e]$. If $V_2 \equiv 1$ and $V_1 \in L_1([1,e]) \setminus L_{\mathcal{B}}([1,e])$, e.g. $V_1(r) = \frac{1}{(r-1)(1+\ln^2(r-1))}$, then the right-hand side of (64) is infinite, while the right-hand side of (55) is finite. Hence (64) does not imply (54), (55). If $V_2 \equiv 1$ and $V_1 \in L_{\mathcal{B}}([1,e]) \setminus \cup_{p>1} L_p([1,e])$, e.g. $V_1(r) = \frac{1}{(r-1)(1+|\ln(r-1)|^3)}$, then the right-hand side of (49) is infinite, while the right-hand side of (11) is finite. Hence (49) does not imply (11). If $V_1 \equiv 1$ on $[1,e]$ and $V_2 \in L_1(\mathbb{S}) \setminus L_{\mathcal{B}}(\mathbb{S})$, e.g. $V_2(\vartheta) = \frac{1}{|\vartheta|(1+\ln^2|\vartheta|)}$, then then the right-hand side of (52) is infinite, while the right-hand side of (64) is finite. Hence (52) does not imply (64). Finally, if $V_1 \equiv 1$ on $[1,e]$ and $V_2 \in L_{\mathcal{B}}(\mathbb{S}) \setminus \cup_{p>1} L_p(\mathbb{S})$, e.g. $V_2(\vartheta) = \frac{1}{|\vartheta|(1+|\ln|\vartheta||^3)}$, then the right-hand side of (54) is infinite, while the right-hand side of (11) is finite. Hence (54) does not imply (11).

9 Concluding remarks

Using estimate (56), one can prove that Theorem 1.1 and Proposition 1.2 in [21] remain true if one substitute the condition $V_{\mathcal{N}} \in \mathcal{L}_1(\mathbb{R}_+, L_p(\mathbb{S}))$, $p > 1$ with

$$V_{\mathcal{N}} \in \mathcal{L}_1(\mathbb{R}_+, L_{\mathcal{B}}(\mathbb{S}))\,. \tag{72}$$

In particular, if (72) is satisfied, then the Weyl-type asymptotic formula

$$\lim_{\alpha\to+\infty} \alpha^{-1} N_-(\mathcal{E}_{\alpha V}) = \frac{1}{4\pi}\int_{\mathbb{R}^2} V(x)\,dx$$

holds if and only if

$$\lim_{s\to0+} s\,\mathrm{card}\,\{n \in \mathbb{Z} :\ A_n > s\} = 0 \tag{73}$$

(see (39)). One can also prove that, if (72) is satisfied, then $N_-(\mathcal{E}_{\alpha V}) = O(\alpha)$ as $\alpha \to +\infty$ if and only if $\|(A_n)_{n\in\mathbb{Z}}\|_{1,\infty} < \infty$ (cf. [20, Theorem 1.1]). The last result and (52) imply that if

$$\sum_{\{n\in\mathbb{Z}:\ \alpha A_n > c\}} \sqrt{\alpha A_n} = O(\alpha) \quad \text{as} \quad \alpha \to +\infty,$$

then $\|(A_n)_{n\in\mathbb{Z}}\|_{1,\infty} < \infty$. This implication is a special case ($q = 1$, $\sigma = 1/2$) of the following result on sequences of numbers a_n, $n \in \mathbb{Z}$: if $q > \sigma > 0$, then

$$\sum_{\{n\in\mathbb{Z}:\ \alpha|a_n|>c\}} (\alpha|a_n|)^\sigma = O(\alpha^q) \ \text{ as } \ \alpha \to +\infty \tag{74}$$

$$\iff \text{card}\,\{n \geq 0: \ |a_n| > s\} = O(s^{-q}) \ \text{ as } \ s \to 0+ . \tag{75}$$

Indeed, if (75) holds, then one gets similarly to (48)

$$\begin{aligned}
&\sum_{\alpha|a_n|>c} (\alpha|a_n|)^\sigma = \alpha^\sigma \sum_{|a_n|>c/\alpha} \int_0^{|a_n|} \sigma s^{\sigma-1} ds \\
&= \sigma\alpha^\sigma \int_0^\infty s^{\sigma-1}\,\text{card}\,\{n \geq 0: \ |a_n| > s \ \& \ |a_n| > c/\alpha\}\, ds \\
&= \sigma\alpha^\sigma \int_0^{c/\alpha} s^{\sigma-1}\,\text{card}\,\{n \geq 0: \ |a_n| > c/\alpha\}\, ds \\
&+\sigma\alpha^\sigma \int_{c/\alpha}^\infty s^{\sigma-1}\,\text{card}\,\{n \geq 0: \ |a_n| > s\}\, ds \\
&= \alpha^\sigma\, O(\alpha^q) \int_0^{c/\alpha} s^{\sigma-1}\, ds + \alpha^\sigma\, O\left(\int_{c/\alpha}^\infty s^{\sigma-1-q}\, ds\right) \\
&= O(\alpha^q) \ \text{ as } \ \alpha \to +\infty,
\end{aligned}$$

where the last two integrals exist due to the condition $q > \sigma > 0$. Suppose now (74) holds. Then using the notation $\alpha = c/s$ one gets

$$\begin{aligned}
&\text{card}\,\{n \geq 0: \ |a_n| > s\} = \text{card}\,\{n \geq 0: \ |a_n| > c/\alpha\} \\
&\leq \frac{1}{c^\sigma} \sum_{\alpha|a_n|>c} (\alpha|a_n|)^\sigma = O(\alpha^q) = O(s^{-q}) \ \text{ as } \ s \to 0+ .
\end{aligned}$$

Let us return to the discussion of the necessity of (73) and $\|(A_n)_{n\in\mathbb{Z}}\|_{1,\infty} < \infty$. Neither of these conditions is necessary for $N_-(\mathcal{E}_V)$ to be finite. Indeed, let $V \geq 0$ be a radial potential such that $(A_n)_{n\in\mathbb{Z}} \in \ell_\infty(\mathbb{Z}) \setminus \ell_{1,\infty}(\mathbb{Z})$. Then it follows from Theorem 7.1 that $N_-(\mathcal{E}_{\beta V}) = 1$ for sufficiently small $\beta > 0$, although the above conditions are not satisfied for βV. It turns out that the condition $(A_n)_{n\in\mathbb{Z}} \in \ell_\infty(\mathbb{Z})$ on the other hand is necessary for $N_-(\mathcal{E}_V)$ to be finite.

Theorem 9.1. *Let $V \geq 0$. Then*

$$N_-(\mathcal{E}_V) \geq \frac{1}{3}\,\text{card}\,\{n \in \mathbb{Z}: \ A_n \geq 10\pi\}. \tag{76}$$

Proof. Let $m \in \mathbb{Z}$ be such that $A_m \geq 10\pi$ and let $r_0 < r_1 < r_2 < r_3$ be the radii of the boundary circles of the annuli U_{m-1}, U_m, U_{m+1}. Consider the function

$$w_m(x) := \begin{cases} 0, & |x| \leq r_0 \;\text{ or }\; |x| \geq r_3, \\ 1 - \frac{\ln(r_1/|x|)}{\ln(r_1/r_0)}, & r_0 < |x| < r_1, \\ 1, & r_1 \leq |x| \leq r_2, \\ \frac{\ln(r_3/|x|)}{\ln(r_3/r_2)}, & r_2 < |x| < r_3. \end{cases}$$

It is easy to see that

$$\int_{\mathbb{R}^2} |\nabla w_m(x)|^2\, dx = 2\pi \int_{r_0}^{r_1} \frac{1}{(\ln(r_1/r_0))^2}\, \frac{1}{r}\, dr + 2\pi \int_{r_2}^{r_3} \frac{1}{(\ln(r_3/r_2))^2}\, \frac{1}{r}\, dr$$
$$= 2\pi \left(\frac{1}{\ln(r_1/r_0)} + \frac{1}{\ln(r_3/r_2)} \right)$$
$$= \begin{cases} 2\pi \left(2^{-|m|} + 2^{2-|m|}\right), & m \leq -2 \;\text{ or }\; m \geq 2, \\ 2\pi, & m = \pm 1, \\ 4\pi, & m = 0 \end{cases}$$
$$\leq 2\pi \left(2^{-|m|} + 2^{2-|m|}\right) = 10\pi\, 2^{-|m|},$$

and the inequality is strict if $m = 0, \pm 1$. Since $w_m(x) = 1$ for $x \in U_m$, one gets

$$\int_{\mathbb{R}^2} V(x) |w_m(x)|^2\, dx \geq \int_{U_m} V(x) dx$$
$$\geq \begin{cases} \frac{1}{2^{|m|}} \int_{U_m} V(x) |\ln|x||\, dx, & m \neq 0, \\ A_0, & m = 0 \end{cases}$$
$$= \frac{A_m}{2^{|m|}} \geq 10\pi\, 2^{-|m|} \geq \int_{\mathbb{R}^2} |\nabla w_m(x)|^2\, dx,$$

and the second inequality is strict if $m \neq 0$. Hence $\mathcal{E}_V[w_m] < 0$ if $A_m \geq 10\pi$, and (76) follows from the fact that w_m and w_k have disjoint supports if $|m - k| \geq 3$. Indeed, if the set $\Sigma := \{n \in \mathbb{Z} :\; A_n \geq 10\pi\}$ is infinite, then it contains infinitely many elements that lie at a distance at least 3 from each other, and both sides of (76) are infinite. If the set Σ is nonempty and finite, take

$$m_1 := \min \Sigma, \quad m_{j+1} = \min\{n \in \Sigma :\; n \geq m_j + 3\},\; j = 1, \dots$$

and continue the process until the set on the right-hand side is empty. This produces at least $\frac{1}{3}\,\mathrm{card}\,\Sigma$ numbers and concludes the proof. $\square$

Note that none of the estimates in the paper is sharp in the sense that $N_-(\mathcal{E}_V)$ has to be infinite if the right-hand side is infinite. Indeed, the examples at the end of Section 8 show that the right-hand side of (52) may be infinite while $N_-(\mathcal{E}_V)$ is finite due to (64), and the other way around: the right-hand side of (64) may be infinite while $N_-(\mathcal{E}_V)$ is finite due to (52). On the other hand, no estimate of the type

$$N_-(\mathcal{E}_V) \le \mathrm{const} + \int_{\mathbb{R}^2} V(x)W(x)\,dx + \mathrm{const}\,\|V\|_{\Psi,\mathbb{R}^2} \tag{77}$$

can hold with an Orlicz norm $\|\cdot\|_\Psi$ weaker than $\|\cdot\|_{\mathcal{B}}$, provided the weight function W is bounded in a neighborhood of at least one point (cf. [29, Section 4]). Indeed, let Ω be a bounded open set where W is bounded and suppose $\Psi(s)/\mathcal{B}(s) \to 0$ as $s \to \infty$. Let Φ be the complementary function to Ψ. Then it follows from [15] and [18, Lemma 13.1] that there exist $w_j \in \mathring{W}^1_2(\Omega)$, $j \in \mathbb{N}$ such that $\|w\|_{W^1_2(\Omega)} = 1$ and $\|w_j^2\|_{\Phi,\Omega} \to \infty$ as $j \to \infty$. The Banach-Steinhaus theorem and [18, Theorem 14.2] imply the existence of $v \in L_\Psi(\Omega)$ such that the sequence

$$\int_\Omega v(x)w_j^2(x)\,dx, \quad j \in \mathbb{N}$$

is unbounded. Define $V \in L_\Psi(\mathbb{R}^2)$ by $V(x) = |v(x)|$ for $x \in \Omega$ and $V(x) = 0$ for $x \notin \Omega$. Then

$$\mathcal{Q}(V) := \sup_{\|w\|_{W^1_2(\mathbb{R}^2)}=1} \int_{\mathbb{R}^2} V(x)|w(x)|^2\,dx = \infty.$$

Since the quadratic form $\int_{\mathbb{R}^2} V(x)|w(x)|^2dx \ge 0$ is closable in $W^1_2(\mathbb{R}^2)$, it corresponds to a nonnegative self-adjoint operator. Since $\mathcal{Q}(V) = \infty$, the operator is unbounded, and it follows from the spectral theorem that there exists an infinite-dimensional subspace $\mathcal{F} \subset W^1_2(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} V(x)|w(x)|^2dx > \|w\|^2_{W^1_2(\mathbb{R}^2)}, \quad \forall w \in \mathcal{F} \setminus \{0\}.$$

Hence $N_-(\mathcal{E}_V) = \infty$, but the right-hand side of (77) is finite. Below is a more constructive proof of the same result.

Theorem 9.2. *Let $W \geq 0$ be bounded in a neighborhood of at least one point and let Ψ be an N-function such that*

$$\lim_{s\to\infty} \frac{\Psi(s)}{\mathcal{B}(s)} = 0.$$

Then there exists a compactly supported $V \geq 0$ such that

$$\int_{\mathbb{R}^2} V(x)W(x)\,dx + \|V\|_{\Psi,\mathbb{R}^2} < \infty$$

and $N_-(\mathcal{E}_V) = \infty$.

Proof. Shifting the independent variable if necessary, we can assume that W is bounded in a neighborhood of 0. Let $r_0 > 0$ be such that W is bounded in the open ball $B(0, r_0)$.
Let

$$\gamma(s) := \sup_{t\geq s} \frac{\Psi(t)}{\mathcal{B}(t)}\,.$$

Then γ is a non-increasing function, $\gamma(s) \to 0$ as $s \to \infty$, and $\Psi(s) \leq \gamma(s)\mathcal{B}(s)$. Since Ψ is an N-function, $\Psi(s)/s \to \infty$ as $s \to \infty$ (see [18, (1.16)]). Hence there exists $s_0 \geq e$ such that $\Psi(s) \geq s$ and $\gamma(s) \leq 1$ for $s \geq s_0$.
Choose $r_k \in (0, 1/s_0)$, $k \in \mathbb{N}$ in such a way that $r_k < \frac{1}{3}\, r_{k-1}$ and

$$\sum_{k=1}^{\infty} \gamma\left(\frac{1}{r_k}\right) < \infty.$$

It is easy to see that the open disks $B(2r_k, r_k)$, $k \in \mathbb{N}$ lie in $B(0, r_0)$ and are pairwise disjoint.
Let

$$t_k := \tfrac{3}{\ln \frac{1}{r_k}}\, r_k^{-4}\,,$$

$$V(x) := \begin{cases} t_k, & x \in B\left(2r_k, r_k^2\right),\ k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$t_k = \frac{3}{\ln \frac{1}{r_k}}\, r_k^{-4} > \frac{1}{r_k^3 \ln \frac{1}{r_k}}\, r_k^{-1} > \frac{1}{r_k}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^2} \Psi(V(x))\, dx &= \sum_{k=1}^{\infty} \pi r_k^4\, \Psi(t_k) \le \pi \sum_{k=1}^{\infty} r_k^4\, \gamma(t_k)\mathcal{B}(t_k) \\
&< \pi \sum_{k=1}^{\infty} r_k^4\, \gamma(t_k)(1+t_k)\ln(1+t_k) < 4\pi \sum_{k=1}^{\infty} r_k^4\, \gamma(t_k)\, t_k \ln t_k \\
&= 4\pi \sum_{k=1}^{\infty} \gamma(t_k)\, \frac{3r_k^4}{r_k^4\, \ln\frac{1}{r_k}}\, \ln \frac{3}{r_k^4\, \ln\frac{1}{r_k}} < 4\pi \sum_{k=1}^{\infty} \gamma(t_k)\, \frac{3}{\ln\frac{1}{r_k}}\, \ln\frac{3}{r_k^4} \\
&< 72\pi \sum_{k=1}^{\infty} \gamma(t_k) \le 72\pi \sum_{k=1}^{\infty} \gamma\left(\frac{1}{r_k}\right) < \infty.
\end{aligned}$$

Hence $\|V\|_{\Psi,\mathbb{R}^2} < \infty$. Since $t_k > 1/r_k \ge s_0$, we have $t_k \le \Psi(t_k)$ and

$$\int_{\mathbb{R}^2} V(x)\, dx \le \int_{\mathbb{R}^2} \Psi(V(x))\, dx < \infty.$$

Taking into account that the support of V lies in $B(0, r_0)$ and that W is bounded in $B(0, r_0)$, we get

$$\int_{\mathbb{R}^2} V(x)W(x)\, dx < +\infty.$$

Let

$$w_k(x) := \begin{cases} 1, & |x - 2r_k| \le r_k^2, \\ \frac{\ln(r_k/|x-2r_k|)}{\ln(1/r_k)}, & r_k^2 < |x - 2r_k| \le r_k, \\ 0, & |x - 2r_k| > r_k \end{cases}$$

(cf. [10]). Then

$$\int_{\mathbb{R}^2} |\nabla w_k(x)|^2\, dx = \frac{2\pi}{\ln(1/r_k)}$$

and

$$\int_{\mathbb{R}^2} V(x)|w_k(x)|^2\, dx \ge \int_{B\left(2r_k, r_k^2\right)} V(x)\, dx = \pi r_k^4\, t_k = \frac{3\pi}{\ln(1/r_k)}\, .$$

Hence

$$\mathcal{E}_V[w_k] < 0, \quad \forall k \in \mathbb{N}$$

and $N_-(\mathcal{E}_V) = \infty$. $\square$

It is probably difficult to obtain an estimate for $N_-(\mathcal{E}_V)$ that is sharp in the above sense, i.e. is such that $N_-(\mathcal{E}_V)$ is infinite if the right-hand side is infinite. Indeed, there are potentials $V \ge 0$ such that $N_-(\mathcal{E}_{\alpha V}) < \infty$ for $\alpha < 1$ and $N_-(\mathcal{E}_{\alpha V}) = \infty$ for $\alpha > 1$. For such potentials, $N_-(\mathcal{E}_V)$ may be finite or infinite and the following theorem shows that, in the latter case, $N_-(\mathcal{E}_{\alpha V})$ may grove arbitrarily fast or arbitrarily slow as $\alpha \to 1-0$.

Theorem 9.3. *i) For any $N \in \mathbb{N}$, there exists a $V \in L_1(\mathbb{R}^2)$, $V \ge 0$, such that $N_-(\mathcal{E}_V) = N$ and $N_-(\mathcal{E}_{\alpha V}) = \infty$ for $\alpha > 1$.*
ii) For any sequence $(\alpha_k)_{k\in\mathbb{N}}$ increasing to 1 and any $N_k \in \mathbb{N}$, there exists a $V \in L_1(\mathbb{R}^2)$, $V \ge 0$, such that $N_k \le N_-(\mathcal{E}_{\alpha_{k+1}V}) - N_-(\mathcal{E}_{\alpha_k V}) < \infty$, $k \in \mathbb{N}$.
iii) For any sequence $(\alpha_k)_{k\in\mathbb{N}}$ increasing to 1 and satisfying the condition

$$\sum_{k=1}^{\infty} \frac{1-\alpha_{k+1}}{\alpha_{k+1}-\alpha_k} < \infty\,, \tag{78}$$

there exist a $V \in L_1(\mathbb{R}^2)$, $V \ge 0$ and a $k_0 \in \mathbb{N}$ such that $N_-(\mathcal{E}_{\alpha_{k+1}V}) - N_-(\mathcal{E}_{\alpha_k V}) = 1$ for all $k \ge k_0$.

Proof. See Appendix B. $\square$

10 Appendix A: Sharp 1-dimensional Sobolev type inequalities

Let $0 < a < b$. It follows from the embedding $W_2^1([a,b]) \hookrightarrow C([a,b])$ that there exist constants $\alpha, \beta > 0$ such that

$$\frac{|u(x)|^2}{|x|} \le \alpha \int_a^b |u'(t)|^2\,dt + \beta \int_a^b \frac{|u(t)|^2}{|t|^2}\,dt, \quad \forall u \in W_2^1([a,b]).$$

This inequality is used in Section 5 and it is natural to ask what the best values of $\alpha, \beta > 0$ are. Since there are two constants involved here, it is convenient to rewrite the inequality in the following form

$$\frac{|u(x)|^2}{|x|} \le C(\kappa)\left(\int_a^b |u'(t)|^2\,dt + \kappa \int_a^b \frac{|u(t)|^2}{|t|^2}\,dt\right), \quad \forall u \in W_2^1([a,b]), \tag{79}$$

and to look for the best value of $C(\kappa)$ for a given $\kappa > 0$.

Lemma 10.1. *Let*

$$\gamma_1 = \frac{-1+\sqrt{1+4\kappa}}{2}\,, \quad \gamma_2 = \frac{-1-\sqrt{1+4\kappa}}{2}\,. \tag{80}$$

For any $\kappa > 0$, (79) holds with

$$C(\kappa;x) := \frac{\gamma_1^2 x^{\sqrt{1+4\kappa}} + \kappa\left(b^{\sqrt{1+4\kappa}} + a^{\sqrt{1+4\kappa}}\right) + \gamma_2^2 (ab)^{\sqrt{1+4\kappa}} x^{-\sqrt{1+4\kappa}}}{\kappa\sqrt{1+4\kappa}\,\left(b^{\sqrt{1+4\kappa}} - a^{\sqrt{1+4\kappa}}\right)}$$

and becomes an equality for

$$u(t) = \begin{cases} \left(\gamma_2 b^{\sqrt{1+4\kappa}} - \gamma_1 x^{\sqrt{1+4\kappa}}\right)\left(\gamma_1 t^{\gamma_1} - \gamma_2 a^{\sqrt{1+4\kappa}} t^{\gamma_2}\right) t, & a \le t < x, \\ \left(\gamma_2 a^{\sqrt{1+4\kappa}} - \gamma_1 x^{\sqrt{1+4\kappa}}\right)\left(\gamma_1 t^{\gamma_1} - \gamma_2 b^{\sqrt{1+4\kappa}} t^{\gamma_2}\right) t, & x < t \le b. \end{cases}$$

The maximum of $C(\kappa;x)$ is achieved at $x = a$ and is equal to

$$C(\kappa) := \frac{1}{2\kappa}\left(1 + \sqrt{1+4\kappa}\,\frac{b^{\sqrt{1+4\kappa}} + a^{\sqrt{1+4\kappa}}}{b^{\sqrt{1+4\kappa}} - a^{\sqrt{1+4\kappa}}}\right). \tag{81}$$

Proof. It is easy to see that

$$u(x)\ln\frac{b}{a} = \int_a^b G_x(t)u'(t)\,dt + \int_a^b \frac{u(t)}{t}\,dt,$$

where

$$G_x(t) := \begin{cases} \ln\frac{t}{a}, & a \le t < x, \\ \ln\frac{t}{b}, & x < t \le b. \end{cases}$$

Since

$$\int_a^b \varphi(t)u'(t)\,dt + \int_a^b (t\varphi'(t))\,\frac{u(t)}{t}\,dt = 0, \quad \forall \varphi \in \mathring{W}_2^1\big((a,b)\big),$$

we also have

$$u(x)\ln\frac{b}{a} = \int_a^b (G_x(t) - \varphi(t))u'(t)\,dt + \int_a^b (1 - t\varphi'(t))\,\frac{u(t)}{t}\,dt,$$

and the Cauchy–Schwarz inequality implies

$$\frac{|u(x)|^2}{|x|} \le \frac{\mathcal{J}(\varphi)}{|x|\ln^2\frac{b}{a}}\left(\int_a^b |u'(t)|^2\,dt + \kappa\int_a^b \frac{|u(t)|^2}{|t|^2}\,dt\right),$$

where

$$\mathcal{J}(\varphi) := \int_a^b |G_x(t) - \varphi(t)|^2\,dt + \frac{1}{\kappa}\int_a^b |1 - t\varphi'(t)|^2\,dt.$$

Hence we need to minimise $\mathcal{J}(\varphi)$ on $\mathring{W}_2^1\big((a,b)\big)$. It is clear we only need to consider real-valued φ. The Euler equation for this functional takes the form

$$t^2\varphi'' + 2t\varphi' - \kappa\varphi = 1 - \kappa G_x, \quad \varphi(a) = 0 = \varphi(b).$$

It is easy to solve the equation on (a,x) and separately on (x,b): the function $\varphi_0 = G_x$ is a solution on both intervals, and the change of the independent variable $s = \ln t$ reduces the corresponding homogeneous equation to an ODE with constant coefficients. Choosing the constants in the general solutions on (a,x) and (x,b) in such a way that $\varphi(a) = 0$, $\varphi(b) = 0$, $\varphi(x-0) = \varphi(x+0)$ and $\varphi'(x-0) = \varphi'(x+0)$, one gets

$$\varphi(t) = G_x(t) + \frac{x^{-\gamma_1}\ln\frac{b}{a}}{(\gamma_1-\gamma_2)\left(b^{\sqrt{1+4\kappa}} - a^{\sqrt{1+4\kappa}}\right)}\Phi_x(t), \tag{82}$$

$$\Phi_x(t) := \begin{cases} \left(\gamma_2 b^{\sqrt{1+4\kappa}} - \gamma_1 x^{\sqrt{1+4\kappa}}\right)\left(t^{\gamma_1} - a^{\sqrt{1+4\kappa}}t^{\gamma_2}\right), & a \le t < x, \\ \left(\gamma_2 a^{\sqrt{1+4\kappa}} - \gamma_1 x^{\sqrt{1+4\kappa}}\right)\left(t^{\gamma_1} - b^{\sqrt{1+4\kappa}}t^{\gamma_2}\right), & x < t \le b. \end{cases}$$

Taking into account that γ_1 and γ_2 (see (80)) are the roots of the quadratic equation $\gamma^2 + \gamma - \kappa = 0$, one can easily check that (82) does indeed solve the equation on (a,x) and (x,b) and satisfy the above conditions at $t = a, x, b$.
It follows from the above that (79) holds with $\mathcal{J}(\varphi)/(|x|\ln^2\frac{b}{a})$ in place of $C(\kappa)$ for any $\varphi \in \mathring{W}_2^1\big((a,b)\big)$, in particular for the one given by (82). Let us show that the equality in (79) is achieved for the latter. Indeed, let

$$\begin{aligned} u(t) &:= \frac{1}{\kappa}\left(t - t^2\varphi'(t)\right) \\ &= M(\kappa)\begin{cases} \left(\gamma_2 b^{\sqrt{1+4\kappa}} - \gamma_1 x^{\sqrt{1+4\kappa}}\right)\left(\gamma_1 t^{\gamma_1} - \gamma_2 a^{\sqrt{1+4\kappa}}t^{\gamma_2}\right)t, & a \le t < x, \\ \left(\gamma_2 a^{\sqrt{1+4\kappa}} - \gamma_1 x^{\sqrt{1+4\kappa}}\right)\left(\gamma_1 t^{\gamma_1} - \gamma_2 b^{\sqrt{1+4\kappa}}t^{\gamma_2}\right)t, & x < t \le b, \end{cases} \\ M(\kappa) &:= -\frac{x^{-\gamma_1}\ln\frac{b}{a}}{\kappa(\gamma_1-\gamma_2)\left(b^{\sqrt{1+4\kappa}} - a^{\sqrt{1+4\kappa}}\right)}. \end{aligned}$$

Then $u \in W_2^1([a,b])$, $u' = (1 - t^2\varphi''(t) - 2t\varphi'(t))/\kappa = G_x - \varphi$, and the Cauchy–Schwarz inequality used above is in fact an equality.
It is left to evaluate $\mathcal{J}(\varphi)/(|x|\ln^2\frac{b}{a})$ for (82). Using the equalities

$$\begin{aligned} &2\gamma_1 + 1 = \sqrt{1+4\kappa}\,, \quad 2\gamma_2 + 1 = -\sqrt{1+4\kappa}\,, \\ &1 + \frac{\gamma_1^2}{\kappa} = \frac{\sqrt{1+4\kappa}}{\kappa}\gamma_1, \quad 1 + \frac{\gamma_2^2}{\kappa} = -\frac{\sqrt{1+4\kappa}}{\kappa}\gamma_2, \\ &2\kappa\gamma_1 - \gamma_1^2 = \sqrt{1+4\kappa}\,\gamma_1^2, \quad 2\kappa\gamma_2 - \gamma_2^2 = -\sqrt{1+4\kappa}\,\gamma_2^2, \end{aligned}$$

one gets after a straightforward but not a particularly pleasant calculation

$$\frac{\mathcal{J}(\varphi)}{|x|\ln^2\frac{b}{a}} = \frac{\gamma_1^2 x^{\sqrt{1+4\kappa}} + \kappa\left(b^{\sqrt{1+4\kappa}} + a^{\sqrt{1+4\kappa}}\right) + \gamma_2^2 (ab)^{\sqrt{1+4\kappa}} x^{-\sqrt{1+4\kappa}}}{\kappa\sqrt{1+4\kappa}\,\left(b^{\sqrt{1+4\kappa}} - a^{\sqrt{1+4\kappa}}\right)}.$$

Since the function $z \mapsto \gamma_1^2 z + \gamma_2^2 (ab)^{\sqrt{1+4\kappa}} z^{-1}$ does not have a local maximum for $z > 0$ ($z = x^{\sqrt{1+4\kappa}}$), the above fraction achieves its maximum on $[a, b]$ at an endpoint. It is easy to see that the maximum is achieved at $x = a$ and is equal to (81). $\square$

Remark 10.2. Suppose $u \in W_2^1([0,1])$ and $u(0) = 0$. Then using (79), (81) with $b = 1$ and $a \to 0+$ one gets

$$\frac{|u(x)|^2}{|x|} \le \frac{1}{2\kappa}\left(1 + \sqrt{1+4\kappa}\right)\left(\int_0^1 |u'(t)|^2\, dt + \kappa \int_0^1 \frac{|u(t)|^2}{|t|^2}\, dt\right), \quad \forall x \in (0,1],$$

and the right-hand side is finite due to Hardy's inequality. Note that

$$\frac{1}{2\kappa}\left(1 + \sqrt{1+4\kappa}\right) < C(\kappa)$$

for any $b > a > 0$ (see (81)).

Starting with the representation

$$u(x) = \int_0^1 H_x(t) u'(t)\, dt + \int_0^1 u(t)\, dt,$$

where

$$H_x(t) := \begin{cases} t, & 0 \le t < x, \\ t - 1, & x < t \le 1, \end{cases}$$

one can find, similarly to (79), (81), the optimal constant $C_0(\kappa)$ in the estimate

$$|u(x)|^2 \le C_0(\kappa)\left(\int_0^1 |u'(t)|^2\, dt + \kappa\int_0^1 |u(t)|^2\, dt\right), \quad \forall u \in W_2^1([0,1]).$$

The calculations are easier in this case, and one gets after an affine transformation of the independent variable that for any $\kappa > 0$,

$$|u(x)|^2 \le C_0(\kappa)\left((b-a)\int_a^b |u'(t)|^2\, dt + \frac{\kappa}{b-a}\int_a^b |u(t)|^2\, dt\right), \quad \forall u \in W_2^1([a,b]) \tag{83}$$

holds with

$$C_0(\kappa;x) := \frac{\sinh(2\sqrt{\kappa}) + \sinh\left(2\sqrt{\kappa}\,\frac{x-a}{b-a}\right) + \sinh\left(2\sqrt{\kappa}\,\frac{b-x}{b-a}\right)}{4\sqrt{\kappa}\,\sinh^2\sqrt{\kappa}}$$

and becomes an equality for

$$u(t) = \begin{cases} \cosh\left(\sqrt{\kappa}\,\frac{b-x}{b-a}\right)\,\cosh\left(\sqrt{\kappa}\,\frac{t-a}{b-a}\right), & a \le t < x, \\ \cosh\left(\sqrt{\kappa}\,\frac{x-a}{b-a}\right)\,\cosh\left(\sqrt{\kappa}\,\frac{b-t}{b-a}\right), & x < t \le b. \end{cases}$$

The maximum of $C_0(\kappa;x)$ is achieved at $x = a$ and $x = b$, and is equal to

$$C_0(\kappa) := \frac{\coth\sqrt{\kappa}}{\sqrt{\kappa}}\,. \tag{84}$$

One can rewrite the inequality

$$|u(x)|^2 \le \frac{\coth\sqrt{\kappa}}{\sqrt{\kappa}}\left((b-a)\int_a^b |u'(t)|^2\,dt + \frac{\kappa}{b-a}\int_a^b |u(t)|^2\,dt\right), \tag{85}$$
$$\forall x \in [a,b], \quad \forall u \in W_2^1([a,b])$$

in the following more symmetric form (with $\varrho = \sqrt{\kappa}$):

$$|u(x)|^2 \le \coth\varrho\left(\frac{b-a}{\varrho}\int_a^b |u'(t)|^2\,dt + \frac{\varrho}{b-a}\int_a^b |u(t)|^2\,dt\right), \quad \forall \varrho > 0,$$

and the equality here is achieved for

$$u(t) = \cosh\left(\varrho\,\frac{t-a}{b-a}\right),\ x = b \quad \text{and} \quad u(t) = \cosh\left(\varrho\,\frac{t-b}{b-a}\right),\ x = a.$$

11 Appendix B: Proof of Theorem 9.3

Proof. All potentials V appearing in the proof are radial $V(x) = F(|x|)$ and satisfy the following conditions: $V(x) = 0$ if $|x| \le a_0$, and

$$0 \le V(x) \le \frac{\gamma}{|x|^2\ln^2|x|}\,, \quad |x| > a_0 \tag{86}$$

with certain $a_0, \gamma > 0$. We can take $\gamma = 1/3$ in i) and $\gamma = 1/(3\alpha_1)$ in ii), iii). According to (27),

$$N_-(\mathcal{E}_V) = N_-(\mathcal{E}_{\mathcal{R},V}) + N_-(\mathcal{E}_{\mathcal{N},V})$$

for radial potentials. It follows from the proof of Theorem 7.1 or from Lemma 4.1 in [20] that one can choose $a_0 > 0$ large enough to get $N_-(\mathcal{E}_{\mathcal{N},V}) = 0$. We assume below that $a_0 > 1$ and that the last equality holds. Then

$$N_-(\mathcal{E}_V) = N_-(\mathcal{E}_{X,G}),$$

where $G(t) = e^{2t}F(e^t)$ (see (28) and (30)). The usual approximation argument shows that $N_-(\mathcal{E}_{X,G}) = N_-(\mathcal{E}_{\mathcal{H}_1,G})$, where $\mathcal{E}_{\mathcal{H}_1,G}$ denotes the form (30) with the domain

$$\mathcal{H}_1 := \left\{u \in W^1_{2,\mathrm{loc}}(\mathbb{R}) : \int_{\mathbb{R}} |u'|^2\, dt + \int_{\mathbb{R}} \frac{|u|^2}{1+|t|^2}\, dt < \infty\right\}.$$

Note that (86) is equivalent to

$$0 \le G(t) \le \frac{\gamma}{|t|^2}, \quad t > a_1 := \ln a_0 > 0.$$

We assume throughout the proof that $G(t) = 0$ for $t \le a_1$.
It follows from the above that it is sufficient to prove the theorem with $N_-(\mathcal{E}_{\mathcal{H}_1,\alpha G})$ in place of $N_-(\mathcal{E}_{\alpha V})$.
Let $0 < a < b < \infty$. We denote by $\mathcal{E}_{\mathcal{H}_1(a,b),G}$, $\mathcal{E}^{a,b}_G$ and $\mathcal{E}_{\mathcal{H}_1(b,\infty),G}$ the forms defined by (30) on the domains

$$\begin{aligned}&\mathcal{H}_1(a,b) := \{u \in \mathcal{H}_1 :\ u(a) = 0 = u(b)\}, \quad \mathring{W}^1_2\big((a,b)\big), \quad \text{and}\\ &\mathcal{H}_1(b,\infty) := \{u \in \mathcal{H}_1 :\ u(t) = 0,\ t \le b\}\end{aligned}$$

respectively, and we also use the following notation

$$\mathcal{E}^{a,b}_\beta[u] := \int_a^b |u'(t)|^2 dt - \beta \int_a^b \frac{|u(t)|^2}{t^2}\, dt, \quad \mathrm{Dom}\,(\mathcal{E}^{a,b}_\beta) = \mathring{W}^1_2\big((a,b)\big). \tag{87}$$

i) Let $G(t) = 1/(3t^2)$ for $a_1 < t \le a_2$ and $G(t) = 1/(4t^2)$ for $t > a_2$, where a_2 is chosen in such a way that

$$\ln\frac{a_2}{a_1} = \sqrt{3}\left(2N - \frac{3}{2}\right)\pi, \quad \text{i.e.} \quad \frac{1}{3} = \frac{1}{4} + \left(\frac{(N-3/4)\pi}{\ln\frac{a_2}{a_1}}\right)^2.$$

Let $\alpha \in (3/4, 1)$. We will need two auxiliary functions u_1 and u_2 that solve the equation $u'' + \alpha G u = 0$ on $(-\infty, a_1)$, (a_1, a_2), and $(a_2, +\infty)$. On each of these intervals, G has the form const/t^2, and the change of the independent

variable $s = \ln t$ reduces the equation to an ODE with constant coefficients. Let

$$\mu_1 := \sqrt{\frac{\alpha}{3} - \frac{1}{4}}, \quad \mu_2 := \frac{1 - \sqrt{1-\alpha}}{2},$$

$$u_1(t) := \begin{cases} \rho a_1^{1/2} \sin(\mu_1 \ln a_1 + \delta), & t < a_1, \\ \rho t^{1/2} \sin(\mu_1 \ln t + \delta), & a_1 < t < a_2, \\ t^{\mu_2}, & t > a_2, \end{cases}$$

$$u_2(t) := \begin{cases} 0, & t < a_1, \\ t^{1/2} \sin(\mu_1 (\ln t - \ln a_1)), & a_1 < t < a_2, \\ a_2^{1/2-\mu_2} \sin\left(\mu_1 \ln \frac{a_2}{a_1}\right) t^{\mu_2}, & t > a_2, \end{cases}$$

where the constants $\rho, \delta \in \mathbb{R}$ are chosen in such a way that $u_1(a_2 - 0) = u_1(a_2 + 0)$, $u_1'(a_2 - 0) = u_1'(a_2 + 0)$. It is easy to see that $u_1 \in \mathcal{H}_1$ and $u_2 \in \mathcal{H}_1(a_1, \infty)$ are indeed solutions on the above intervals and that $u_1(a_1 - 0) = u_1(a_1 + 0)$, $u_2(a_2 - 0) = u_2(a_2 + 0)$. Integration by parts gives the following

$$\begin{aligned}
&\mathcal{E}_{\mathcal{H}_1,\alpha G}(u_1, u) = 0, \quad \forall u \in \mathcal{H}_1(a_1), \quad \text{in particular,} \quad \mathcal{E}_{\mathcal{H}_1,\alpha G}(u_1, u_2) = 0, \\
&\mathcal{E}_{\mathcal{H}_1,\alpha G}(u_2, u) = 0, \quad \forall u \in \mathcal{H}_1(a_1, a_2), \\
&\mathcal{E}_{\mathcal{H}_1,\alpha G}[u_1] = u_1(a_1)(u_1'(a_1 - 0) - u_1'(a_1 + 0)) \\
&\quad = -\rho^2 \sin^2(\mu_1 \ln a_1 + \delta) \left(\frac{1}{2} + \mu_1 \cot(\mu_1 \ln a_1 + \delta)\right), \\
&\mathcal{E}_{\mathcal{H}_1,\alpha G}[u_2] = u_1(a_2)(u_2'(a_2 - 0) - u_2'(a_2 + 0)) \\
&\quad = \sin^2\left(\mu_1 \ln \frac{a_2}{a_1}\right) \left(\frac{1}{2} - \mu_2 + \mu_1 \cot\left(\mu_1 \ln \frac{a_2}{a_1}\right)\right).
\end{aligned}$$

If α is close to 1, then $\mu_1 \ln \frac{a_2}{a_1}$ is close to

$$\frac{1}{2\sqrt{3}} \ln \frac{a_2}{a_1} = \left(N - \frac{3}{4}\right) \pi,$$

and hence $\mathcal{E}_{\mathcal{H}_1,\alpha G}[u_2] > 0$. The condition

$$\frac{u_1'(a_2 - 0)}{u_1(a_2 - 0)} = \frac{u_1'(a_2 + 0)}{u_1(a_2 + 0)}$$

is equivalent to $1/2 + \mu_1 \cot(\mu_1 \ln a_2 + \delta) = \mu_2$. Again, if α is close to 1, then $\mu_2 < 1/2$ is close to $1/2$. Hence $\cot(\mu_1 \ln a_2 + \delta)$ is a small negative number,

i.e. $\mu_1 \ln a_2 + \delta = (m + 1/2)\pi + \epsilon$, where ϵ is a small positive number and $m \in \mathbb{Z}$. Consequently,

$$\mu_1 \ln a_1 + \delta = (m + 1/2)\pi + \epsilon - \mu_1 \ln \frac{a_2}{a_1}$$

is close to $(m - N + 1 + 1/4)\pi + \epsilon$, and $\mathcal{E}_{\mathcal{H}_1, \alpha G}[u_1] < 0$.
It is easy to see that any $u \in \mathcal{H}_1$ admits a unique representation

$$u = d_1 u_1 + d_2 u_2 + u_0, \quad d_1, d_2 \in \mathbb{C},\ u_0 \in \mathcal{H}_1(a_1, a_2),$$

and it follows from the above that

$$\mathcal{E}_{\mathcal{H}_1, \alpha G}[u] = |d_1|^2 \mathcal{E}_{\mathcal{H}_1, \alpha G}[u_1] + |d_2|^2 \mathcal{E}_{\mathcal{H}_1, \alpha G}[u_2] + \mathcal{E}_{\mathcal{H}_1, \alpha G}[u_0].$$

Since $\mathcal{E}_{\mathcal{H}_1, \alpha G}[u_1] < 0$ and $\mathcal{E}_{\mathcal{H}_1, \alpha G}[u_2] > 0$, we have

$$\begin{aligned}
&N_-(\mathcal{E}_{\mathcal{H}_1, \alpha G}) = 1 + N_-(\mathcal{E}_{\mathcal{H}_1(a_1, a_2), \alpha G}) \\
&= 1 + N_-\left(\mathcal{E}^{a_1, a_2}_{\alpha/3}\right) + N_-\left(\mathcal{E}_{\mathcal{H}_1(a_2, \infty), \alpha G}\right) = 1 + (N - 1) + 0 = N,
\end{aligned}$$

where the penultimate equality follows from Lemma 11.1 (see below) and Hardy's inequality (see, e.g., [14, Theorem 327]). Hence

$$N_-(\mathcal{E}_{\mathcal{H}_1, G}) = \lim_{\alpha \to 1-0} N_-(\mathcal{E}_{\mathcal{H}_1, \alpha G}) = N.$$

On the other hand,

$$N_-(\mathcal{E}_{\mathcal{H}_1, \alpha G}) \ge N_-\left(\mathcal{E}_{\mathcal{H}_1(a_2, \infty), \alpha G}\right) = +\infty, \quad \forall \alpha > 1,$$

where the last equality holds because $\alpha G > 1/4$ on (a_2, ∞) (see, e.g., Lemma 11.1).

ii) Take any $\alpha_0 \in (0, \alpha_1)$, $N_0 = 0$, and define $a_2, a_3, \dots$ successively by

$$\sqrt{\frac{\alpha_k}{2(\alpha_k + \alpha_{k-1})} - \frac{1}{4}} \ln \frac{a_{k+1}}{a_k} = \left(N_{k-1} + 2 + \frac{1}{2}\right)\pi, \quad k \in \mathbb{N}.$$

Let

$$G(t) := \frac{1}{2(\alpha_k + \alpha_{k-1})\, t^2}, \quad a_k < t < a_{k+1},\ k \in \mathbb{N}.$$

Then

$$\begin{aligned}
&N_-(\mathcal{E}_{\mathcal{H}_1, \alpha_k G}) \le 2 + N_-(\mathcal{E}_{\mathcal{H}_1(a_1, a_{k+1}), \alpha G}) \\
&= 2 + N_-\left(\mathcal{E}^{a_1, a_{k+1}}_{\alpha_k G}\right) + N_-\left(\mathcal{E}_{\mathcal{H}_1(a_{k+1}, \infty), \alpha_k G}\right) = 2 + N_-\left(\mathcal{E}^{a_1, a_{k+1}}_{\alpha_k G}\right)
\end{aligned}$$

due to Hardy's inequality, and

$$\begin{aligned} N_-(\mathcal{E}_{\mathcal{H}_1,\alpha_{k+1}G}) &\geq N_-\left(\mathcal{E}^{a_1,a_{k+1}}_{\alpha_{k+1}G}\right) + N_-\left(\mathcal{E}^{a_{k+1},a_{k+2}}_{\alpha_{k+1}G}\right) \\ &\geq N_-\left(\mathcal{E}^{a_1,a_{k+1}}_{\alpha_k G}\right) + N_k + 2 \geq N_-(\mathcal{E}_{\mathcal{H}_1,\alpha_k G}) + N_k \end{aligned}$$

due to Lemma 11.1.

iii) Take $k_0 \in \mathbb{N}$ such that

$$\sum_{k \geq k_0 - 1} \frac{1-\alpha_{k+1}}{\alpha_{k+1}-\alpha_k} \leq \frac{1}{8} \tag{88}$$

and set $\beta_k := \alpha_{k_0-1+k}$, $k = 0, 1, \ldots,$

$$\begin{aligned} \mu_{1,k,j} &:= \sqrt{\frac{\beta_k}{2(\beta_j+\beta_{j-1})} - \frac{1}{4}}\,, \\ \mu_{1,k} &:= \mu_{1,k,k} = \frac{1}{2}\sqrt{\frac{\beta_k-\beta_{k-1}}{\beta_k+\beta_{k-1}}}\,, \\ \mu_{2,k} &:= \frac{1-\sqrt{1-\frac{2\beta_k}{\beta_{k+1}+\beta_k}}}{2} = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\beta_{k+1}-\beta_k}{\beta_{k+1}+\beta_k}}\,, \\ \epsilon_k &:= \frac{8\pi}{9}\,\frac{1-\beta_k}{\beta_k-\beta_{k-1}} < \frac{\pi}{9}\,, \quad k \in \mathbb{N},\ j = 1, \ldots, k. \end{aligned} \tag{89}$$

Define $a_2, a_3, \ldots$ successively by

$$\mu_{1,k} \ln \frac{a_{k+1}}{a_k} = \pi + \epsilon_k, \quad k \in \mathbb{N}.$$

Then

$$\begin{aligned} &\pi + \epsilon_j \leq \mu_{1,k,j} \ln \frac{a_{j+1}}{a_j} < \sqrt{\frac{1}{2(\beta_j+\beta_{j-1})} - \frac{1}{4}}\, \ln \frac{a_{j+1}}{a_j} \\ &= \sqrt{\frac{2-\beta_j-\beta_{j-1}}{\beta_j-\beta_{j-1}}}\,(\pi+\epsilon_j) = \sqrt{1+2\frac{1-\beta_j}{\beta_j-\beta_{j-1}}}\,(\pi+\epsilon_j) \\ &\leq \left(1+\frac{1-\beta_j}{\beta_j-\beta_{j-1}}\right)(\pi+\epsilon_j) = \pi+\epsilon_j+\pi\,\frac{1-\beta_j}{\beta_j-\beta_{j-1}} \\ &+\epsilon_j\,\frac{1-\beta_j}{\beta_j-\beta_{j-1}} < \pi + 2\pi\,\frac{1-\beta_j}{\beta_j-\beta_{j-1}}\,, \quad j = 1, \ldots, k. \end{aligned} \tag{90}$$

Let

$$G(t) := \frac{1}{2(\beta_k + \beta_{k-1})\, t^2}\,, \quad a_k < t < a_{k+1},\ k \in \mathbb{N}.$$

Similarly to part i) of the proof, define

$$u_{1,k}(t) := \begin{cases} \rho_{1,1} a_1^{1/2} \sin(\mu_{1,k,1} \ln a_1 + \delta_{1,1}), & t < a_1, \\ \\ \rho_{1,j} t^{1/2} \sin(\mu_{1,k,j} \ln t + \delta_{1,j}), & a_j < t < a_{j+1}, \\ & j = 1, \ldots, k, \\ \\ t^{\mu_{2,k}}, & t > a_{k+1}, \end{cases}$$

$$u_{2,k}(t) := \begin{cases} 0, & t < a_1, \\ \\ t^{1/2} \sin(\mu_{1,k,1}(\ln t - \ln a_1)), & a_1 < t < a_2, \\ \\ \rho_{2,j} t^{1/2} \sin(\mu_{1,k,j} \ln t + \delta_{2,j}), & a_j < t < a_{j+1}, \\ & j = 2, \ldots, k, \\ \\ \rho_{2,k} a_{k+1}^{1/2-\mu_{2,k}} \sin\left(\mu_{1,k} \ln a_{k+1} + \delta_{2,k}\right)\, t^{\mu_{2,k}}, & t > a_{k+1}, \end{cases}$$

where the constants $\rho_{l,j}, \delta_{l,j} \in \mathbb{R}$, $l = 1, 2$ are chosen in such a way that

$$\begin{aligned} &u_{1,k}(a_j - 0) = u_{1,k}(a_j + 0),\ u'_{1,k}(a_j - 0) = u'_{1,k}(a_j + 0),\ j = 2, \ldots, k+1, \\ &u_{2,k}(a_j - 0) = u_{2,k}(a_j + 0),\ u'_{2,k}(a_j - 0) = u'_{2,k}(a_j + 0),\ j = 2, \ldots, k. \end{aligned}$$

If $k = 1$, one needs to define $u_{2,k}$ in a slightly different way, which is closer to the definition in part i):

$$u_{2,1}(t) := \begin{cases} 0, & t < a_1, \\ t^{1/2} \sin(\mu_{1,1}(\ln t - \ln a_1)), & a_1 < t < a_2, \\ a_2^{1/2-\mu_{2,1}} \sin\left(\mu_{1,1} \ln \frac{a_2}{a_1}\right) t^{\mu_{2,1}}, & t > a_2. \end{cases}$$

It is easy to see that $u_{1,k} \in \mathcal{H}_1$ and $u_{2,k} \in \mathcal{H}_1(a_1, \infty)$ are solutions of the equation $u'' + \beta_k G u = 0$ on the intervals $(-\infty, a_1)$, (a_j, a_{j+1}), $j = 1, \ldots, k$, and $(a_{k+1}, +\infty)$, and that $u_{1,k}(a_1 - 0) = u_{1,k}(a_1 + 0)$, $u_{2,k}(a_{k+1} - 0) = u_{2,k}(a_{k+1} + 0)$. Exactly as in part i), any $u \in \mathcal{H}_1$ admits a unique representation

$$u = d_1 u_{1,k} + d_2 u_{2,k} + u_0, \quad d_1, d_2 \in \mathbb{C},\ u_0 \in \mathcal{H}_1(a_1, a_{k+1}),$$

and one has

$$\mathcal{E}_{\mathcal{H}_1, \beta_k G}[u] = |d_1|^2 \mathcal{E}_{\mathcal{H}_1, \beta_k G}[u_{1,k}] + |d_2|^2 \mathcal{E}_{\mathcal{H}_1, \beta_k G}[u_{2,k}] + \mathcal{E}_{\mathcal{H}_1, \beta_k G}[u_0], \tag{91}$$

$$\mathcal{E}_{\mathcal{H}_1,\beta_k G}[u_{1,k}] =$$
$$-\rho_{1,1}^2 \sin^2(\mu_{1,k,1}\ln a_1 + \delta_{1,1}) \left(\frac{1}{2} + \mu_{1,k,1}\cot(\mu_{1,k,1}\ln a_1 + \delta_{1,1})\right),$$
$$\mathcal{E}_{\mathcal{H}_1,\beta_k G}[u_{2,k}] = \rho_{2,k}^2 \sin^2(\mu_{1,k}\ln a_{k+1} + \delta_{2,k})\Big(\frac{1}{2} - \mu_{2,k}$$
$$+\mu_{1,k}\cot(\mu_{1,k}\ln a_{k+1} + \delta_{2,k})\Big), \quad k > 1,$$
$$\mathcal{E}_{\mathcal{H}_1,\beta_1 G}[u_{2,1}] =$$
$$\sin^2\left(\mu_{1,1}\ln\frac{a_2}{a_1}\right)\left(\frac{1}{2} - \mu_{2,1} + \mu_{1,1}\cot\left(\mu_{1,1}\ln\frac{a_2}{a_1}\right)\right).$$

Similarly to part i), it follows from (89) that $\mathcal{E}_{\mathcal{H}_1,\beta_1 G}[u_{2,1}] > 0$. Let us show that $\mathcal{E}_{\mathcal{H}_1,\beta_k G}[u_{2,k}] > 0$ for $k > 1$ as well. It follows from (88) and (90) that $u_{2,k}$ has exactly one zero in (a_1, a_2) and that

$$\pi + \epsilon_1 < \mu_{1,k,1}\ln\frac{a_2}{a_1} < \pi + 2\pi\,\frac{1-\beta_1}{\beta_j - \beta_0} < \pi + \frac{\pi}{4}.$$

The condition
$$\frac{u_{2,k}'(a_2 - 0)}{u_{2,k}(a_2 - 0)} = \frac{u_{2,k}'(a_2 + 0)}{u_{2,k}(a_2 + 0)}$$
is equivalent to

$$\mu_{1,k,1}\cot\left(\mu_{1,k,1}\ln\frac{a_2}{a_1}\right) = \mu_{1,k,2}\cot(\mu_{1,k,2}\ln a_2 + \delta_{2,2}).$$

Since $\mu_{1,k,1} > \mu_{1,k,2}$, we get

$$\pi < \mu_{1,k,2}\ln a_2 + \delta_{2,2} + m\pi < \mu_{1,k,1}\ln\frac{a_2}{a_1} < \pi + 2\pi\,\frac{1-\beta_1}{\beta_1 - \beta_0}$$

for some $m \in \mathbb{Z}$. Using (88) and (90) again, we see that $u_{2,k}$ has exactly one zero in (a_2, a_3) and that

$$2\pi < \mu_{1,k,2}\ln a_3 + \delta_{2,2} + m\pi < 2\pi + 2\pi\left(\frac{1-\beta_1}{\beta_1-\beta_0} + \frac{1-\beta_2}{\beta_2-\beta_1}\right) < 2\pi + \frac{\pi}{4}.$$

Continuing the above argument, we show that $u_{2,k}$ has exactly one zero in each interval (a_j, a_{j+1}), $j = 1, \ldots, k$, and that

$$k\pi < \mu_{1,k}\ln a_{k+1} + \delta_{2,k} + n\pi < k\pi + 2\pi\sum_{j=1}^{k}\frac{1-\beta_j}{\beta_j - \beta_{j-1}} < k\pi + \frac{\pi}{4}$$

for some $n \in \mathbb{Z}$. This inequality implies that $\mathcal{E}_{\mathcal{H}_1,\beta_k G}[u_{2,k}] > 0$.
Our next task is to show that $\mathcal{E}_{\mathcal{H}_1,\beta_k G}[u_{1,k}] < 0$. Suppose the contrary: $\mathcal{E}_{\mathcal{H}_1,\beta_k G}[u_{1,k}] \ge 0$. Then there exists $\ell \in \mathbb{Z}$ such that

$$\omega_k := \operatorname{arccot}\left(-\frac{1}{2\mu_{1,k,1}}\right) \le \mu_{1,k,1}\ln a_1 + \delta_{1,1} + \ell\pi \le \pi.$$

It follows from (88) and (90) that

$$\omega_k + \pi < \mu_{1,k,1}\ln a_2 + \delta_{1,1} + \ell\pi < 2\pi + 2\pi\,\frac{1-\beta_1}{\beta_j - \beta_0}\,,$$

and one shows as above that

$$\omega_k + \pi < \mu_{1,k,2}\ln a_2 + \delta_{1,2} + m\pi < 2\pi + 2\pi\,\frac{1-\beta_1}{\beta_j - \beta_0}$$

for some $m \in \mathbb{Z}$. Continuing as above, one gets

$$\omega_k + k\pi < \mu_{1,k}\ln a_{k+1} + \delta_{1,k} + n\pi < (k+1)\pi + 2\pi\sum_{j=1}^{k}\frac{1-\beta_j}{\beta_j - \beta_{j-1}} < (k+1)\pi + \frac{\pi}{4}$$

for some $n \in \mathbb{Z}$. Then

$$\begin{aligned}&\text{either}\quad \frac{1}{2} + \mu_{1,k}\cot(\mu_{1,k}\ln a_{k+1} + \delta_{1,k}) < 0\\ &\text{or}\quad \frac{1}{2} + \mu_{1,k}\cot(\mu_{1,k}\ln a_{k+1} + \delta_{1,k}) > \frac{1}{2}\,.\end{aligned}$$

On the other hand, the condition

$$\frac{u_{1,k}'(a_{k+1}-0)}{u_{1,k}(a_{k+1}-0)} = \frac{u_{1,k}'(a_{k+1}+0)}{u_{1,k}(a_{k+1}+0)}$$

is equivalent to

$$\frac{1}{2} + \mu_{1,k}\cot(\mu_{1,k}\ln a_{k+1} + \delta_{1,k}) = \mu_{2,k} \in (0, 1/2).$$

The obtained contradiction shows that $\mathcal{E}_{\mathcal{H}_1,\beta_k G}[u_{1,k}] < 0$. Now it follows from (91) that

$$\begin{aligned}&N_-(\mathcal{E}_{\mathcal{H}_1,\beta_k G}) = 1 + N_-(\mathcal{E}_{\mathcal{H}_1(a_1,a_{k+1}),\beta_k G})\\ &= 1 + N_-\left(\mathcal{E}^{a_1,a_{k+1}}_{\beta_k G}\right) + N_-\left(\mathcal{E}_{\mathcal{H}_1(a_{k+1},\infty),\beta_k G}\right) = 1 + N_-\left(\mathcal{E}^{a_1,a_{k+1}}_{\beta_k G}\right),\end{aligned}$$

where the last equality follows from Hardy's inequality. It is easy to see that

$$N_-\left(\mathcal{E}_{\beta_k G}^{a_1,a_{k+1}}\right) \geq \sum_{j=1}^{k} N_-\left(\mathcal{E}_{\beta_k G}^{a_j,a_{j+1}}\right) = k$$

(see (90) and Lemma 11.1). If $N_-\left(\mathcal{E}_{\beta_k G}^{a_1,a_{k+1}}\right) > k$, then there exists $\lambda < 0$ and a nontrivial solution $u \in \mathring{W}_2^1\big((a_1, a_{k+1})\big)$ of $u'' + (\beta_k G + \lambda)u = 0$ that has k zeros in (a_1, a_{k+1}) (see, e.g., [22, Ch. I, Theorem 3.3] or [32, Theorem 13.2]). Then $u_{2,k}$ has to have at least $k+1$ zeros in (a_1, a_{k+1}) (see [22, Ch. I, Theorem 3.1] or [32, Theorem 13.3]). On the other hand, we have shown that $u_{2,k}$ has exactly k zeros in (a_1, a_{k+1}). Hence $N_-\left(\mathcal{E}_{\beta_k G}^{a_1,a_{k+1}}\right)$ cannot be larger than k, i.e. $N_-\left(\mathcal{E}_{\beta_k G}^{a_1,a_{k+1}}\right) = k$, and $N_-(\mathcal{E}_{\mathcal{H}_1,\beta_k G}) = k+1$. Finally,

$$\begin{aligned} N_-(\mathcal{E}_{\mathcal{H}_1,\alpha_{k+1} G}) - N_-(\mathcal{E}_{\mathcal{H}_1,\alpha_k G}) = N_-(\mathcal{E}_{\mathcal{H}_1,\beta_{k-k_0+2} G}) - N_-(\mathcal{E}_{\mathcal{H}_1,\beta_{k-k_0+1} G}) \\ = 1, \quad \forall k \geq k_0. \end{aligned}$$

□

Lemma 11.1. *The following equality holds for the form* (87)

$$N_-\left(\mathcal{E}_\beta^{a,b}\right) = \begin{cases} 0, & \beta \leq \frac{1}{4} + \left(\frac{\pi}{\ln\frac{b}{a}}\right)^2, \\ \\ N, & \frac{1}{4} + \left(\frac{N\pi}{\ln\frac{b}{a}}\right)^2 < \beta \leq \frac{1}{4} + \left(\frac{(N+1)\pi}{\ln\frac{b}{a}}\right)^2, \quad N \in \mathbb{N}. \end{cases}$$

Proof. $\mathcal{E}_\beta^{a,b}$ is the quadratic form of the self-adjoint operator $A_\beta := -\frac{d^2}{dt^2} - \frac{\beta}{t^2}$ on $L_2([a,b])$ with the domain $W_2^2([a,b]) \cap \mathring{W}_2^1\big((a,b)\big)$. The spectrum of A_β is discrete and consists of simple eigenvalues. It follows from Hardy's inequality that the eigenvalues are positive for $\beta \leq 1/4$. As β increases, the eigenvalues move continuously (see, e.g., [16, Theorem V.4.10]) to the left, and $N_-\left(\mathcal{E}_\beta^{a,b}\right)$ increases by one when an eigenvalue crosses 0. This happens for those values of β for which 0 is an eigenvalue of A_β, i.e. when

$$-u'' - \frac{\beta}{t^2}\,u = 0, \quad u(a) = 0 = u(b), \quad \left(\beta > \frac{1}{4}\right)$$

has a nontrivial solution. The change of the independent variable $s = \ln t$ reduces this equation to an ODE with constant coefficients, and one finds that the solutions of the original equation that satisfy the condition $u(a) = 0$ are multiples of

$$t^{1/2} \sin\left(\sqrt{\beta - \frac{1}{4}}\,\big(\ln t - \ln a\big)\right).$$

The latter satisfies the condition $u(b) = 0$ if and only if

$$\sqrt{\beta - \frac{1}{4}} \ln \frac{b}{a} = N\pi, \quad \text{i.e.} \quad \beta = \frac{1}{4} + \left(\frac{N\pi}{\ln \frac{b}{a}}\right)^2, \quad N \in \mathbb{N}.$$

$\square$

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